Weak Laws of Large Numbers of Cesaro Summation for Random Arrays

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Abstract: In this paper, we establish weak laws of large numbers with or without random indices for Cesaro summation for random arrays of random elements in Banach spaces. Our results are more general and stronger than some well-known ones. AMS Subject classification 2000: 60B11, 60B12, 60F05, 60G42.

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1. Introduction

Consider a double array \( \{X_{mn}; m \geq 1, n \geq 1\} \) of random elements defined on a probability space \((\Omega, \mathcal{F}, P)\) taking values in a real separable Banach space \(E\) with norm \(\|\cdot\|\). Let \(\{u_n; n \geq 1\}\) and \(\{v_n; n \geq 1\}\) be sequences of positive integers, let \(\{T_n; n \geq 1\}\) and \(\{\tau_n; n \geq 1\}\) be sequences of positive integer-valued random variables. In the current work, we extend weak laws of large numbers of Cesaro summation for random arrays and for double arrays with random indices.

Limit theorems for weighted sums (with or without random indices) for random variables (real-valued or Banach space-valued) are studied by many authors (see, e.g., Wei and Taylor [1], OrdonezCabrera [2], Adler et al. [3], Sung et al. [4]). Recently, Dung [5] obtained the weak law of large numbers with random indices for double arrays of random elements. In this paper, we establish the weak laws of large numbers with or without random indices for Cesaro summation for random arrays of random elements in a p-uniformly smooth Banach space.

2. Preliminaries

For \(\alpha > -1\), we let
\[
A_n^\alpha = \frac{(\alpha + 1)(\alpha + 2)\ldots(\alpha + n)}{n!}, \quad n = 1, 2, 3, \ldots \quad \text{and} \quad A_0^\alpha = 1.
\]

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Then the definition of Cesaro summability for array is extended as follows.

**Definition 1.** Let \( \alpha, \beta > 0 \). The array \( \{x_{m,n} : m, n \geq 0\} \) is said to be \((C, \alpha, \beta)\)-summable iff

\[
\frac{1}{A_m^\alpha A_n^\beta} \sum_{k,l=0}^{m,n} A_{n-l}^{\alpha-1} A_{m-k}^{\beta-1} x_{kl} \text{ converges as } m, n \to \infty.
\]

It is easy to see that \((C, 1, 1)\)-convergence is the same as convergence of array of the arithmetic mean.

Here we collect some facts that will be used on and off in general without specific reference.

Firstly, we have

\[
A_n^\alpha - \frac{n^\alpha}{\Gamma(\alpha + 1)} \quad \text{as} \quad n \to \infty \quad (1.1)
\]

Secondly, we use the fact that if \( \{a_k, k \geq 1\} \) is a sequence of numbers such that \( A_n \uparrow \infty \) as \( n \to \infty \) where \( A_n = \sum_{k=1}^{n} a_k \) and \( x_n \to 0 \) as \( n \to \infty \) then

\[
\frac{1}{A_n} \sum_{k=1}^{n} a_k x_k \to 0 \quad \text{as} \quad n \to \infty. \quad (1.2)
\]

For \( a, b \in \mathbb{R} \), \( \min\{a, b\} \) and \( \max\{a, b\} \) will be denoted, respectively, by \( a \wedge b, a \vee b \). Throughout this paper, the symbol \( C \) will denote a generic constant \( (0 < C < \infty) \) which is not necessarily the same one in each appearance.

Technical definitions that are relevant to the current work will be discussed in this section. Scalora [6] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element \( X \) and sub-\( \sigma \)-algebra \( G \) of \( \mathcal{F} \), the conditional expectation \( E(X \mid G) \) is defined analogously to that in the random variable case and enjoys similar properties.

A real separable Banach space \( E \) is said to be \( p \)-uniformly smooth \((1 \leq p \leq 2)\) if there exists a finite positive constant \( C \) such that for all martingales \( \{S_n : n \geq 1\} \) with values in \( E \),

\[
\sup_{n \geq 1} E \|S_n\|_p^p \leq C \sum_{n=1}^{\infty} E \|S_n - S_{n-1}\|_p^p.
\]

It can be shown by using classical methods from martingale theory that if \( E \) is \( p \)-uniformly smooth, then for each \( 1 \leq r < \infty \) there exists a finite constant \( C \) such that

\[
E \sup_{n \geq 1} \|S_n\|_r^r \leq C E \left( \sum_{n=1}^{\infty} \|S_n - S_{n-1}\|_p^p \right)^{\frac{r}{p}}.
\]

Clearly, every real separable Banach space is 1-uniformly smooth and the real line (the same as any Hilbert space) is 2-uniformly smooth.

It follows from the Hoffmann-Jøgensen and Pisier [7] that if a Banach space is \( p \)-uniformly smooth, then it is of Rademacher type \( p \). But the notion of \( p \)-uniformly smooth is only superficially similar to that of Rademacher type \( p \) and has a geometric characterization in terms of smoothness.
Let \( \mathcal{F}_{kl} \) be the \( \sigma \)-field generated by the family of random elements \( \{X_{ij}: i < k \text{ or } j < l\} \), \( \mathcal{F}_{1,1} = \{\emptyset; \Omega\}. \) The following lemma which is due to Dung [5] establishes a maximal inequality for double sums of random elements in martingale type \( p \) Banach spaces.

**Lemma 2.** Let \( 1 < p \leq 2 \). Let \( \{X_{ij}: 1 \leq i \leq m, 1 \leq j \leq n\} \) be a collection of \( mn \) random elements in a real separable Banach space such that \( E(X_{ij} | \mathcal{F}_k) = 0 \) for all \( 1 \leq i \leq m, 1 \leq j \leq n \). Then,

\[
E \max_{1 \leq i,j \leq n} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} \right)^p \leq C \sum_{i=1}^{m} \sum_{j=1}^{n} E \|X_{ij}\|^p
\]

(1.3)

where the constant \( C \) is independent of \( m \) and \( n \).

Random elements \( \{X_{mn}: m \geq 1, n \geq 1\} \) are said to be **stochastically dominated** by a random element \( X \) if for some finite constant \( D \)

\[
P(\|X_{mn} - X\| > t) \leq D P(\|DX\| > t), \quad t \geq 0, m \geq 1, n \geq 1.
\]

### 3. The main results

Let \( \{X_{mn}: m \geq 1, n \geq 1\} \) be an array of random elements defined on a probability space \( (\Omega, \mathcal{F}, P) \) and taking values in a real separable Banach space \( \mathbb{E} \) with norm \( \|\cdot\| \), \( \mathcal{F}_{kl} \) be a \( \sigma \)-field generated by \( \{X_{ij}: i < k \text{ or } j < l\} \), \( \mathcal{F}_{1,1} = \{\emptyset; \Omega\} \). Let \( \{u_n: n \geq 1\}, \{v_n: n \geq 1\} \) be sequences of positive integers such that \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = \infty \). For any set \( A \), we denote \( I(A) \) the indicator function, i.e,

\[
I(A)(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}
\]

Set

\[
Y_{kl}^{mn} = A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} X_{ij} I(A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} \|X_{ij}\| \leq A_{m}^{\alpha} A_{n}^{\beta})
\]

where \( \alpha, \beta > 0 \)

**Theorem 3.** Let \( 1 \leq p \leq 2 \), \( \alpha, \beta > 0 \) and \( \mathbb{E} \) be a \( p \)-uniformly smooth Banach space. Suppose that

\[
\sum_{i=1}^{u_n} \sum_{j=1}^{v_n} P(A_{m}^{\alpha-1} A_{n}^{\beta-1} \|X_{ij}\| > A_{m}^{\alpha} A_{n}^{\beta}) \to 0 \quad \text{as } m \vee n \to \infty
\]

and

\[
\frac{1}{(A_{m}^{\alpha} A_{n}^{\beta})^p} \sum_{i=1}^{u_n} \sum_{j=1}^{v_n} E \|Y_{ij}^{mn} - E(Y_{ij}^{mn} | G_{ij})\|^p \to 0 \quad \text{as } m \vee n \to \infty.
\]

(2.2)

Then

\[
\max_{1 \leq i \leq u_n, 1 \leq j \leq v_n} \frac{1}{A_{m}^{\alpha} A_{n}^{\beta}} \sum_{i=1}^{u_n} \sum_{j=1}^{v_n} A_{m-i}^{\alpha-1} A_{n-j}^{\beta-1} Y_{ij} - E(Y_{ij}^{mn} | G_{ij}) \| \to 0 \quad \text{as } m \vee n \to \infty.
\]

(2.3)
Proof. For an arbitrary $\varepsilon > 0$,

$$P\left\{ \max_{i \leq k \leq n, j = 1} \frac{1}{A_{m_{i, j}}^{\alpha_{i, j}}} \left\| \sum_{i = 1}^{k} \sum_{j = 1}^{l} A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \left( \frac{Y_{ij}^{mn} - E(Y_{ij}^{mn} | G_{ij})}{G_{ij}} \right) \right\| > \varepsilon \right\}$$

$$\leq P\left\{ \max_{i \leq k \leq n, j = 1} \frac{1}{A_{m_{i, j}}^{\alpha_{i, j}}} \left\| \sum_{i = 1}^{k} \sum_{j = 1}^{l} A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} X_{ij} - E(Y_{ij}^{mn} | G_{ij}) \right\| > \varepsilon / 2 \right\}$$

$$+ P\left\{ \max_{i \leq k \leq n, j = 1} \frac{1}{A_{m_{i, j}}^{\alpha_{i, j}}} \left\| \sum_{i = 1}^{k} \sum_{j = 1}^{l} \left( Y_{ij}^{mn} - E(Y_{ij}^{mn} | G_{ij}) \right) \right\| > \varepsilon / 2 \right\}$$

$$\leq P\left\{ \bigcup_{i = 1}^{m} \bigcup_{j = 1}^{n} \left( A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \left( X_{ij} > A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \right) \right) \right\}$$

$$+ P\left\{ \max_{i \leq k \leq n, j = 1} \frac{1}{A_{m_{i, j}}^{\alpha_{i, j}}} \left\| \sum_{i = 1}^{k} \sum_{j = 1}^{l} \left( Y_{ij}^{mn} - E(Y_{ij}^{mn} | G_{ij}) \right) \right\| > \varepsilon / 2 \right\}$$

$$\leq \sum_{i = 1}^{m} \sum_{j = 1}^{n} P\left( A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \left( X_{ij} > A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \right) \right)$$

$$+ \frac{2^{p}}{\varepsilon^{p} \left( A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \right)^{p}} E \left( \max_{i \leq k \leq n, j = 1} \left\| \sum_{i = 1}^{k} \sum_{j = 1}^{l} \left( Y_{ij}^{mn} - E(Y_{ij}^{mn} | G_{ij}) \right) \right\|^{p} \right)$$

(by Markov’s inequality)

$$\leq \sum_{i = 1}^{m} \sum_{j = 1}^{n} P\left( A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \left( X_{ij} > A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \right) \right)$$

$$+ \frac{C}{\left( A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \right)^{p}} \sum_{i = 1}^{m} \sum_{j = 1}^{n} E \left( \left\| Y_{ij}^{mn} - E(Y_{ij}^{mn} | G_{ij}) \right\|^{p} \right)$$

(by Lemma 2)

$$\rightarrow 0 \text{ as } m \vee n \rightarrow \infty \text{ (by (3.1) and (3.2)).}$$

The proof is completed. \hfill \square

Corollary 4. Let $1 \leq p \leq 2$, $\alpha > 0$, $\beta > 0$ and $E$ be a $p$-uniformly smooth Banach space. If

$$\sum_{i = 1}^{m} \sum_{j = 1}^{n} P\left( A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \left( X_{ij} > A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} \right) \right) \rightarrow \infty \text{ as } m \vee n \rightarrow \infty,$$

$$\max_{i \leq k \leq n, j = 1} \frac{1}{A_{m_{i, j}}^{\alpha_{i, j}}} \left\| \sum_{i = 1}^{k} \sum_{j = 1}^{l} A_{m_{i, j}}^{\alpha_{i, j}} A_{n_{i, j}}^{\beta_{i, j}} E(Y_{ij}^{mn} | G_{ij}) \right\|^{p} \rightarrow 0 \text{ as } m \vee n \rightarrow \infty$$

and
\[
\frac{1}{(A^\alpha A^\beta)^p} \sum_{i=1}^{m_n} \sum_{j=1}^{v_n} \|E(Y_{ij}^m | G_{ij})\|^p \to 0 \quad \text{as} \quad m \lor n \to \infty, \tag{2.4}
\]

then
\[
\max_{1 \leq k \leq m_n, \ 1 \leq \ell \leq v_n} \frac{1}{A^\alpha A^\beta} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} A^{\alpha-1} A^{\beta-1} X_{ij} \right\|^p \to 0 \quad \text{as} \quad m \lor n \to \infty. \tag{2.5}
\]

**Remark 5.** If the condition (3.4) is replaced by the condition that
\[
\frac{1}{A^\alpha A^\beta} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} A^{\alpha-1} A^{\beta-1} E(Y_{ij}^m | G_{ij}) \right\|^p \to 0 \quad \text{as} \quad m \lor n \to \infty,
\]

then the conclusion (3.5) will be replaced by
\[
\frac{1}{A^\alpha A^\beta} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} A^{\alpha-1} A^{\beta-1} X_{ij} \right\|^p \to 0 \quad \text{as} \quad m \lor n \to \infty.
\]

The following result is a random index version of Theorem 3.

**Theorem 6.** Let \(1 \leq p \leq 2, \ \alpha, \beta > 0\) and \(E\) be a \(p\)-uniformly smooth Banach space. Suppose that \(\{T_n \geq 1\} \ \text{and} \ \{\tau_n \geq 1\} \) are sequences of positive integer-valued random variables such that
\[
\lim_{n \to \infty} n P(T_n > u_n) = \lim_{n \to \infty} n P(\tau_n > v_n) = 0. \tag{2.6}
\]

If
\[
\frac{1}{(A^\alpha A^\beta)^p} \sum_{i=1}^{u_n} \sum_{j=1}^{v_n} P(A^{\alpha-1} A^{\beta-1} \| X_{ij} \| > A^\alpha A^\beta) \to \infty \quad \text{as} \quad m \lor n \to \infty,
\]

and
\[
\frac{1}{(A^\alpha A^\beta)^p} \sum_{i=1}^{u_n} \sum_{j=1}^{v_n} E \| Y_{ij}^m - E(Y_{ij}^m | G_{ij}) \|^p \to 0 \quad \text{as} \quad m \lor n \to \infty,
\]

then
\[
\max_{1 \leq k \leq m_n, \ 1 \leq \ell \leq v_n} \frac{1}{A^\alpha A^\beta} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} A^{\alpha-1} A^{\beta-1} (X_{ij} - E(Y_{ij}^m | G_{ij})) \right\|^p \to 0 \quad \text{as} \quad m \lor n \to \infty.
\]

**Proof.** For arbitrary \(\epsilon > 0\),
\[ P \left( \max_{1 \leq k \leq \infty, m \leq n} \frac{1}{A_n^{\alpha} A_n^{\beta}} \left\| \sum_{i=1}^{\infty} \sum_{j=1}^{l} A_{m-1}^{\alpha-1} A_{n-j}^{\beta-1} (X_{ij} - E(Y_{ij}^{mn} | G_{ij})) \right\| > \epsilon \right) \]

\[ \leq P \left( \max_{1 \leq k \leq \infty, m \leq n} \frac{1}{A_n^{\alpha} A_n^{\beta}} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} A_{m-1}^{\alpha-1} A_{n-j}^{\beta-1} (X_{ij} - E(Y_{ij}^{mn} | G_{ij})) \right\| > \epsilon \right) \cap (T_m \leq u_m) \cap (\tau_n \leq v_n) \]

\[ + P(T_m > u_m) + P(\tau_n > v_n) \]

\[ \rightarrow 0 \text{ as } m \vee n \rightarrow \infty, \text{ (by (3.6) and Theorem 3)} \]

which completes the proof. \[ \square \]

We shall now prove the following extension of the well-known Feller theorem for Cesaro summation for random arrays of random elements in Banach spaces.

**Theorem 7.** Let \( 1 \leq p \leq 2, \ 0 < \alpha < \beta \leq 1, E \) be a \( p \)-uniformly smooth Banach space. Suppose that \( \{X_{mn} : m \geq 1, n \geq 1\} \) is stochastically dominated by a random element \( X \). If

\[ \lim_{n \to \infty} nP[\|X\| > n] = 0, \quad (2.7) \]

then

\[ \max_{1 \leq k \leq \infty, m \leq n} \frac{1}{A_n^{\alpha} A_n^{\beta}} \left\| \sum_{i=1}^{\infty} \sum_{j=1}^{l} A_{m-1}^{\alpha-1} A_{n-j}^{\beta-1} (X_{ij} - E(Y_{ij}^{mn} | G_{ij})) \right\| \to 0 \text{ as } m \vee n \to \infty. \]

**Proof.** We verify (3.1) and (3.2) with \( u_m = m, \ v_n = n \). For (3.1), we have

\[ \sum_{i,j=1}^{m,n} P(A_{i,j}^{\alpha-1} A_{i,j}^{\beta-1} \| X_{ij} \| > A_i^{\alpha} A_j^{\beta}) \leq C \sum_{i,j=1}^{m,n} P(A_{i,j}^{\alpha-1} A_{i,j}^{\beta-1} \| X \| > A_i^{\alpha} A_j^{\beta}) \]

\[ \leq C \sum_{i,j=1}^{m,n} P(i^{\alpha-1} j^{\beta-1} \| X_{ij} \| > m^{\alpha} n^{\beta}) \text{ by (2.1)} \]

\[ = C \frac{1}{m^{\alpha} n^{\beta}} \sum_{i,j=1}^{m,n} i^{\alpha-1} j^{\beta-1} m^{\alpha} m^{\beta} i^{\alpha-1} j^{\beta-1} P(i^{\alpha-1} j^{\beta-1} \| X_{ij} \| > m^{\alpha} n^{\beta}) \to 0 \]

(by (2.2) and (3.7)).

For (3.2), by Jensen’s inequality for conditional expectation, we get

\[ \frac{1}{(A_n^{\alpha} A_n^{\beta})^p} \sum_{i=1}^{m} \sum_{j=1}^{n} E \| Y_{ij}^{mn} - E(Y_{ij}^{mn} | G_{ij}) \|^{p} \leq \frac{C}{(A_n^{\alpha} A_n^{\beta})^p} \sum_{i=1}^{m} \sum_{j=1}^{n} E \| Y_{ij}^{mn} \|^{p} \]
\[
\leq \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m} \sum_{j=1}^{n} E[(i^{p(\alpha-1)} j^{p(\beta-1)}) \| X_{ij} \|^p I(i^{\alpha-1} j^{\beta-1} \| X_{ij} \| \leq m^{\alpha} n^{\beta})]
\]
\[
= \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} \| X_{ij} \|^p I((k-1)^{\alpha} < i^{\alpha-1} j^{\beta-1} \| X_{ij} \| \leq k^{\alpha})
\]
\[
\leq \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} k^{\beta p} P((k-1)^{\alpha} < i^{\alpha-1} j^{\beta-1} \| X \| \leq k^{\alpha})
\]
\[
= \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} k^{\beta p} (P(i^{\alpha-1} j^{\beta-1} \| X \| > (k-1)^{\alpha}) - P(i^{\alpha-1} j^{\beta-1} \| X \| > k^{\alpha}))
\]
\[
\leq \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} k^{\beta p} P(i^{\alpha-1} j^{\beta-1} \| X \| \geq k^{\alpha})
\]
\[
\leq \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} \frac{1}{m^{\alpha-\alpha} n^{\beta-\alpha}} \sum_{k=1}^{m_{ij}} k^{\beta p-\alpha} (k^{\alpha-1} j^{\beta-1} P(\| X \| \geq k^{\alpha} j^{\beta-1}))
\]
\[
\rightarrow 0 \text{ as } m, n \rightarrow \infty.
\]

By applying Theorem 3, the proof is completed. \(\square\)

The following result is a random index version of Theorem 7.

**Theorem 8.** Let \(1 \leq p \leq 2, 0 < \alpha, \beta \leq 1, \mathbb{E}\) be a \(p\)-uniformly smooth Banach space. Suppose that \(\{X_m, m \geq 1, n \geq 1\}\) is stochastically dominated by a random element \(X\). Suppose that \(\{T_n, n \geq 1\}\) and \(\{\tau_n, n \geq 1\}\) are sequences of positive integer-valued random variables such that

\[
\lim_{n \to \infty} P[T_n > n] = \lim_{n \to \infty} P[\tau_n > n] = 0. \tag{3.6}
\]

If

\[
\lim_{n \to \infty} nP(\| X \| > n) = 0,
\]

then

\[
\max_{1 \leq i \leq T_n, \tau_n \leq l} \left\{ \frac{1}{A_m^\alpha A_n^\beta} \left\| \sum_{i=1}^{l} \sum_{j=1}^{l} A_{i-j} A_{i-j+1} (X_{ij} - E(Y_{ij} | G_j)) \right\|^p \rightarrow 0 \right. \text{ as } m, n \rightarrow \infty.
\]

**Proof.** By the same argument in the proof of Theorem 7 and using Theorem 6. \(\square\)

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References


