ON ALGEBRAIC CHARACTERIZATIONS OF GENERALIZED RIGHT INVERTIBLE OPERATORS IN LINEAR SPACES

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In this paper we introduce a class of generalized right invertible operators. This class contains all right invertible operators and some well-known operators in Analysis as Projection, Differ- enation, Difference and some classes of algebraic operators. We demonstrate that many fundamental properties from the theory of right invertible operators can be extended to generalized right invertible operators as Taylor’s and Taylor-Gontcharov’s formulae and that we apply these results to solve some corresponding equations in generalized right invertible operators.

1. FUNDAMENTAL PROPERTIES OF GENERALIZED RIGHT INVERTIBLE OPERATORS

Let $X$ be a linear space over a field of scalars $F$. Denote by $L(X)$ the set of all linear operators with domains and ranges in $X$ and write: $L_0(X) = \{ A \in L(X) : \text{dom} A = X \}$. The set of all invertible operators in $L(X)$ will be denoted by $R(X)$. For a $D \in R(X)$ we denote, respectively, by $R_D$ and $F_D$ the set of all right inverses and the set of all initial operators for $D$, i.e.

$$ R_D = \{ R \in L_0(X) : DR = I \}, $$

$$ F_D = \{ F \in L_0(X) : F^2 = F, \text{ } FX = \ker D \}. $$

Definition 1. An operator $V \in L(X)$ is said to be generalized invertible ($GI$-operator) if there exists a $W \in L_0(X)$ such that

$$ VWV = V \text{ on dom } V. $$

The set of all $GI$-operators in $L(X)$ will be denoted by $W(X)$. For a $V \in W(X)$ we denote by $W_V$ the set of all generalized inverses in $L_0(X)$ of $V$.

Definition 2. An operator $V \in W(X)$ is said to be possessing a right invertibility of degree $r \in \mathbb{N}$ (shortly: $V$ is right invertible of degree $r$) if there is a $W \in W_V$ such that

$$ \text{Im} (VW - I) \subset \ker V^r, $$

where we admit $V^0 = I$ for the case $r = 0$.

The set of all right invertible operators (in $L(X)$) of degree $r$ will be denoted by $R_r(X)$. Hence, by Definitions 1 and 2, it follows

$$ R(X) = R_0(X) \subset R_1(X) \subset \cdots \subset R_n(X) \subset W(X); \ n = 0, 1, 2, \ldots $$
Position 3. Every $V \in R_1(X)$ is called a generalized right invertible operator (shortly: GR-operator). For a $V \in R_1(X)$ we denote by $R_V^*$ the set of all generalized right inverses $y$: GR-inverses) of $V$. Moreover, if there is $W \in R_V^*$ such that $\text{Im} W \subset \ker (VW - I)$ then $V$ is an almost right inverse of $V$. In that case, we write: $W \in R_V^*$.

Position 1. Let $D \in R(X)$, $R \in R_D$ and let $V = R^m D^n$, where $n \geq m$, $n, m \in N$. Then $V$ is an almost right inverse of $V$.

Write: $W_0 = R^{n-m}$, where we admit $R^0 = I$ for the case $n = m$. Since $R \in L_0(X)$ we get that $W_0 \in L_0(X)$. Using equalities $D^k R^k = I$, we find

$$V^2 W_0 = R^m D^n R^m D^n R^{n-m} = R^m D^n R^{m-n} D^n = R^m D^n V$$ (1)

$$VW_0 V = R^m D^n R^{n-m} R^m D^n = R^m D^n = V.$$ (2)

Using (1) we get $\text{Im} (VW_0 - I) \subset \ker V$. This and (2) together imply that $V \in R^1_V$.

On the other hand, for the case $n \geq 2m$ we have

$$VW_0^2 = R^m D^n R^{n-m} R^{n-m} = R^m D^n R^{m-n} R^m R^{n-2m} = W_0,$$

$W_0 \subset \ker (VW_0 - I)$. Hence $W_0 \in R^1_V$.

Example 1. Let $X = C([0, 1], F)$ and let $D = \frac{d}{dt}$, $(Rx)(t) = \int_0^t x(s)ds$ and $(Fz)(t) = z(t_0) \in [0, 1]$. Write: $V = FD$, then $V \not= 0$ and $V^2 = 0$. It is easy to see that $V \in W(X)$. However, $V \not\in R_r(X)$ for $r \in \{0, 1\}$. Indeed, $V \not\in R(X)$, i.e. $V \not\in R_0(X)$. Suppose that there is $W \in L_0(X)$ such that $\text{Im} (VW - I) \subset \ker V$. We find $V = V^2 W = 0$, which contradicts the condition $V \not= 0$.

Position 2. Let $V \in R_1(X)$ and let $W \in R^1_V$. Then

$$V^n W^m = \begin{cases} V^{n-1} & \text{if } n > m \geq 1, \\ VW & \text{if } n = m, \\ VW^{m-n+1} & \text{if } m > n \geq 1. \end{cases}$$

If $n = m \geq 2$, we find

$$V^m W^m = V^{n-2} (V W^2 W^{n-1} = V^{n-1} W^{n-1} = \ldots = V^2 W^2 = (V^2 W) W = VW.$$

If $m \geq 1$, we have the equalities

$$V^n W^m = V^{n-m} (V^m W^m) = V^{n-m} (VW) = V^{n-m-1} (V^2 W) = V^{n-m}.$$ Finally, for $m > n \geq 1$, we get

$$V^m W^m = (V^m W^m) W^{m-n} = (VW) W^{m-n} = VW^{m-n}.$$

Proof is complete.

Position 3. Let $V \in R_1(X)$ and $W \in R^1_V$. Then $V^n \in R_1(X)$ for all $n \in N^+$ and $W^n \in R^1_V$. 49
Proof. The assumptions and Proposition 2 together imply the following equalities

\[ V^n = (VWV)^{n-1} = VWV^n = V^2WV^n = V^2W^2V^n = \ldots = V^nW^nV^n, \]

i.e. \( V^n \in W(X) \). On the other hand, also by Proposition 2, we have

\[ V^n = V^{n-1}VW = V^n(VW) = V^n(V^2W) = V^nV^2W^2 = \ldots = \frac{V^nV^n}{W^n}. \]

Hence \( V^n (V^nW^n - I) = 0 \), which shows that \( V^n \in R_1(X) \) and \( W^n \in R_V \).

**Theorem 1.** Let \( V \in R_1(X) \) and let \( W_0 \in R^1_V \). Then \( W \in L_0(X) \) is an GR-inverse of \( V \) if only if there is an \( A \in L_0(X) \) such that \( \text{Im } A \subset \ker V^2 \) and

\[ W = W_0 + A - W_0VAVW_0. \]

Proof. Let \( W \) is of the form (1), where \( \text{Im } A \subset \ker V^2 \) and \( W_0 \in R^1_V \). We have

\[ VWV = V + VAV - VW_0VAVW_0V = V + VAV - VAV = V, \]

\[ V^2W = V^2W_1 + V^2A - V^2W_1VAVW_1 = V - V^2A - V^2W_1AVW_1 = V. \]

Equalities (5) and (6) together imply \( W \in R^1_V \).

Conversely, let \( W_0, W \in R^1_V \). Write: \( A = W - W_0 \). We find \( V^2A = V^2W - V^2W \)

\[ V - V = 0. \]

i.e. \( \text{Im } A \subset \ker V^2 \). The equalities \( VWV = V \) and \( VW_0V = V \) together in

\[ V(W - W_0)V = 0. \]

Hence, we have

\[ W_0 + A - W_0VAW_0 = W_0 + (W - W_0) - W_0V(W - W_0)VW_0 = W, \]

which gives the representation (1).

**Theorem 2.** Let \( A, B \in L_0(X) \) be given. Then \( I + AB \in R_1(X) \) if and only if \( I + BA \in R_1(X) \).

Moreover, if \( W_{AB} \in R^1_{I+AB} \), then

\[ W = I - BW_{AB}A \in R^1_{I+AB}. \]

Proof. Let \( I + AB \in R_1(X) \) and \( W_{AB} \in R^1_{I+AB} \). Then \( (I + AB)^2W_{AB} = (I + AB) \) and \( W \) def by the formula (1) are well-defined on \( X \). We have the following equalities.

\[ (I + BA)W(I + BA) = (I + BA)(I - BW_{AB}A)(I + BA) \]

\[ = (I + BA)^2 - (I + BA)BW_{AB}A(I + BA) \]

\[ = (I + BA)^2 - B(I + AB)W_{AB}(I + AB)A \]

\[ = (I + AB)^2 - B(I + AB)A = I + BA, \]

which show that \( I + BA \in W(X) \).

On the other hand

\[ (I + BA)^2W_{BA} = (I + BA)^2(I - BW_{AB}A) = (I + BA)^2 - (I + BA)^2BW_{AB}A \]

\[ = (I + BA)^2 - B(I + AB)^2W_{AB}A = (I + BA)^2 - B(I + AB)A = I + B. \]
uply that $W \in R^1_{+F,A}$ and $I + BA \in R_1(X)$. The proof is complete.

**Theorem 4.** Let $V \in R_1(X)$ and $W_0 \in R^1_V$. Write: $W_1 = W_0 V W_0$. Then $W_1 \in R^1_V$ and $W_1 = W_1$.

The assumptions $V^2 W_0 = V$ and $V W_0 V = V$ together imply the following equalities

$$V^2 W_1 = V^2 W_0 V W_0 = V (V W_0 V) W_0 = V^2 W_0 = V,$$

$$W_1 V W_1 = W_0 V W_0 V W_0 V W_0 = W_0 (V W_0 V) W_0 V W_0 = W_0 V W_0 V W_0 = V,$$

$W_1 \in R^1_V$ and $W_1 V W_1 = W_1$.

In sequel, we write: $R^{(1,0)}_V = \{W \in R^1_V : W V W = W\}$.

2. **RIGHT AND LEFT INITIAL OPERATORS FOR GENERALIZED RIGHT INVERTIBLE OPERATORS**

$V \in R_1(X)$, $W \in R^{(1,0)}_V$ and let dim ker $V \neq 0$.

**Theorem 4.** Let $V \in R_1(X)$ and let $W \in R^{(0,1)}_V$. An operator $F_r = F^{(r)}_W \in L_0(X)$ is said to be a right initial operator (shortly: RI operator) for $V$ corresponding to $W$ if $F_r^2 = F_r$, Im $F_r = \ker V$ $\forall \alpha = 0$ on $X$. The set of all RI-operators for $V \in R_1(X)$ will be denoted by $F^{(r)}_V$.

**Theorem 5.** Let $V \in R_1(X)$ and $W \in R^{(1,0)}_V$. Then the operator $F^{(I)}_W = I - V W$ is said to be a left initial operator for $V$ corresponding to $W$.

Note that, if $V \in R(X)$ then $F^{(I)}_W = 0$ for all $W \in R_V$.

**Theorem 5.** Let $V \in R_1(X)$ and let $W \in R^1_V$. Then

$$\text{dom} V = W V (\text{dom} V) \oplus \ker V.$$

Note that for every $x \in \text{dom} V$ we have $W V x \in W V (\text{dom} V)$ and $(I - W V) x \in \ker V$ $W V x + (I - W V) x$. On the other hand, if $x \in W V (\text{dom} V) \cap \ker V$, then there is $V$ such that $x = W V v$ and $V x = 0$, simultaneously. Hence $0 = V x = V (W V) v = V v W (V v) = 0$.

**Theorem 6.** Let $F$ be a RI-operator for $V \in R_1(X)$ corresponding to $W \in R^{(1,0)}_V$. Then $F v = v$ for all $v \in \ker V$.

$V F = 0$.

$X = \ker F \oplus \ker V$.

For every $v \in \ker V$, by Definition 4, there is $z \in \text{dom} V$ such that $v = V z$. Hence $z = V z = v$.

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(ii) Also by Definition 4, for every $x \in X$, we have $F x \in \ker V$. Hence $V F x = 0$, i.e. on $X$.

(iii) It is easy to see that $F \in R_1(X)$ and $F \in R_1^r$. Hence, by Proposition 5, $X = F (F X) \oplus \ker F = \ker V \oplus \ker F$.

**Theorem 3.** Let $V \in R_1(X)$ and let $W \in R_1^{(1,0)}$. Then $F \in L_0(X)$ is an RI-opera\-tor corresponding to $W$ if and only if $\text{Im} F \subset \ker V$ and $F = I - W V$ on $\text{dom} V$.

**Proof.** Let $F \in F^{(r)}_V$ be an RI-operator for $V$ corresponding to $W \in R_1^{(1,0)}$. For any $x \in W$ we find $V x = V W V x = V u$, where $u = W V x$. Hence $x - u \in \ker V$ and by Proposition get $(I - W V) x = x - u = F (x - u) - F x - F u = F x - F W V x = F x$ for $F W = 0$.

Conversely, if $F \in L_0(X)$ such that $\text{Im} F \subset \ker V$ and $F = I - W V$ on $\text{dom} V$, for any $x \in \ker V$ we get $F x = (I - W V) x = z - W V z = z$. Hence $F X = \ker V$ and for $x \in X$, we find $F x = F (F x)$, i.e. $F^2 = F$. On the other hand, since $\text{Im} W \subset \text{dom} V$, $F W = (I - W V) W = W - W V W = 0$. Thus, $F \in F^{(r)}_V$.

**Theorem 4 (Taylor–Gontcharov’s formula).** Let $V \in R_1(X)$ and let $\{F_\gamma \}_{\gamma \in \Gamma}$ be a family $\{W_\gamma \}_{\gamma \in \Gamma}$ of GB-inverses of $V$. Then

$$I = F_\gamma + \sum_{k=0}^{N-1} W_{\gamma_0} \cdots W_{\gamma_k} F_{\gamma_k} V^k + W_{\gamma_0} \cdots W_{\gamma_N} V^N \text{ on } \text{dom} V^N.$$

**Proof.** By induction, for $N = 1$ we have

$$I = F_\gamma - W_{\gamma_0} V + W_{\gamma_0} V = F_\gamma + W_{\gamma_0} V.$$

Suppose that (1) is valid for every $n \leq N$. Then for $n = N + 1$, we find

$$W_{\gamma_0} \cdots W_{\gamma_N} V^{N+1} = W_{\gamma_0} \cdots W_{\gamma_N} (I - F_{\gamma_N}) V^N$$

$$= W_{\gamma_0} \cdots W_{\gamma_N} V^N - W_{\gamma_0} \cdots W_{\gamma_{N-1}} F_{\gamma_N} V^N$$

$$= I - F_{\gamma_0} - \sum_{k=0}^{N-1} W_{\gamma_0} \cdots W_{\gamma_k-1} F_{\gamma_k} V^k - W_{\gamma_0} \cdots W_{\gamma_{N-1}} F_{\gamma_N} V^N$$

$$= I - F_{\gamma_0} - \sum_{k=0}^{N} W_{\gamma_0} \cdots W_{\gamma_k-1} F_{\gamma_k} V^k,$$

which proves (1).

3. **ON GENERALIZED RIGHT INVERTIBILITY OF ALGEBRAIC ELEMENTS**

Let $F = C$, we say that $A \in L_0(X)$ is algebraic if there exists a non-zero normed poly

$$P(t) = t^n + a_1 t^{n-1} + \cdots + a_n$$

with coefficients in $F$ such that $P(A) = 0$ on $X$. An a\-op\-erator $A$ is called of order $n$ if there does not exist a normed polynomial $Q(t)$ of degree such that $Q(A) = 0$ on $X$. Such a minimal polynomial $P(t)$ is called the characteristic poly

of $A$ and is denoted by $P_A(t)$. The set of all algebraic operators in $L_0(X)$ will be denoted by $A_0$.

Let $F = C$ and let $S$ be an algebraic operator in $L_0(X)$ with the characteristic poly

the form

$$P_S(t) = t^N + p_1 t^{N-1} + \cdots + p_{N-1} t + p_N,$$
rem 5. Let $S$ be an algebraic operator of order $N$ in $L_0(X)$ with the characteristic polynomial of the form (9). Then $S \in R_1(X)$ if and only if $p_{N-1}^2 + p_N^2 \neq 0$.

Let $S \in R_1(X)$ and let $W \in R_S^1$. Suppose that $p_{N-1} = 0$ and $P_N = 0$. Since $P_S(S) = 0$, $W = S$, we have the following equalities

$$0 = P_<(S)W = (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-2} I) S^2 W$$

$$= (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-2} I) S$$

$$= S^{N-1} + p_1 S^{N-2} + \cdots + p_{N-2} S,$$

which contradicts the assumption that $S$ is of order $N$.

Conversely, if $P_N \neq 0$, then $S$ is invertible and it is right invertible and $GR$-invertible, simultaneously. We deal with the case when $p_N = 0$ and $p_{N-1} \neq 0$, simultaneously.

Write:

$$W := p_{N-1}^{-2} \left( \sum_{k=0}^{N-2} p_{N-2-p_k} S^{N-k-1} - \sum_{k=0}^{N-3} p_{N-1-p_k} S^{N-k-2} \right).$$

Now check that $W \in R_S^1$. We have the following equalities

$$SW = p_{N-1}^{-2} \left( \sum_{k=0}^{N-2} p_{N-2-p_k} S^{N-k+1} - \sum_{k=0}^{N-3} p_{N-1-p_k} S^{N-k} \right)$$

$$= p_{N-1}^{-2} \left( p_{N-2} S \sum_{k=0}^{N-2} p_k S^{N-k} - p_{N-1} \sum_{k=0}^{N-3} p_k S^{N-k} \right)$$

$$= p_{N-1}^{-2} \left( p_{N-2} (S - p_{N-1} S) - p_{N-1} \sum_{k=0}^{N-3} p_k S^{N-k} \right)$$

$$= p_{N-1}^{-2} \left( - p_{N-1} p_{N-2} S^2 - p_{N-1} \sum_{k=0}^{N-3} p_k S^{N-k} \right)$$

$$= -p_{N-1}^{-1} (S - p_{N-1} S) = S,$$

$\in W(X)$ and $W \in W_S$. On the other hand, we also have $SW = W S$, which gives $S^2 W = S$. The proof is complete.

rem 6. Let $S \in A(X) \cap R_1(X)$. Then there is a unique $W \in R_S^1$.

Let $S$ be of the form (9). By Theorem 5, form the assumptions, we have $p_N^2 + p_{N-1}^2 \neq 0$ and $W \in R_S^1$. If $p_N \neq 0$, then $S$ is invertible. Let $W \in R_S^1$ be arbitrary. Since $S = S$, we find $W = S^{-2} S^2 W = S^{-2} S = S^{-1}$. So that $W = S^{-1}$, i.e. $S$ is uniquely defined. We now deal with the case $p_N = 0$ and $p_{N-1} \neq 0$, simultaneously. Write: $P(t) = p_1 t^{N-3} + \cdots + p_{N-2} t + p_{N-1}$. Then $S = -p_{N-1}^{-1} P(S) S^2$. Let $W \in R_S^1$ be arbitrary. We find

$$SW^2 = -p_{N-1}^{-1} P(S) S^2 W^2 = -p_{N-1}^{-1} (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-3} S + p_{N-2} S W).$$

On the other hand, the equality $S^2 W = S$ follows $SW = -p_{N-1}^{-1} P(S) S$, which gives $W$ of the form

$$W = -p_{N-1}^{-1} (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-3} S + p_{N-2} P(S) S).$$
Thus, $W$ is uniquely determined.

**Corollary 2.** Let $S \in R_1(X)$ and let $W \in R_S^{1(1)}$. Then $S \in \mathcal{A}(X)$ if and only if $W \in \mathcal{A}(X)$.

**Proof.** Let $S \in \mathcal{A}(X)$. Then by Theorem 6, $W$ is uniquely determined as a polynomial in $S$ coefficients in $F$. So that $W \in \mathcal{A}(X)$ ([4]). Conversely, suppose that $W \in \mathcal{A}(X)$ and $P_W t^M + a_1 t^{M-1} + \ldots + a_m$. Then we find $S^{M+1} P_W(W) = 0$ and $S + a_1 S^2 + \ldots + a_M S^M + \ldots$ which gives $S \in \mathcal{A}(X)$.

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**ĐẶC TRUNG ĐẠI SỐ CỦA TOÀN TỪ KHÁ NGHỊCH PHẢI SUY RỌNG TRONG KHÔNG GIAN TUYỂN TÍNH**

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Bài báo đề cập đến một lớp các toán tử khả nghịch phải suy rỗng. Lớp các toán tử này gồm tất cả các toán tử khả nghịch phải, một số lớp toán tử quen biết trong giải tích nhị thức chiều, toán tử vi phân, sai phân và một số dạng toán từ đại số. Các kết quả thủy được tổ chức rất nhiều tính chất cơ bản của lý thuyết các toán từ đại số có thể mở rộng cho tập hợp khả nghịch suy rỗng như các công thức khai triền Taylor, Taylor-Goncharow, công thức diện nghiệm,...