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Stability of the Solution Sets of Parametric Generalized Quasiequilibrium Problems

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Abstract: In this paper we establish sufficient conditions for the solution mappings of parametric generalized vector quasiequilibrium problems to have the stability properties such as lower semicontinuity, upper semicontinuity, Hausdorff lower semicontinuity, continuity, Hausdorff continuity and closedness. The results presented in the paper improve and extend the main results of Kimura-Yao [J. Global Optim. 138, (2008) 429-- 443], Kimura-Yao [Taiwanese J. Math., 12, (2008) 649--669] and Anh-Khanh [J. Math. Anal. Appl., 294, (2004) 699--711]. Some examples are given to illustrate our results.

Keywords: Parametric generalized quasiequilibrium problems, lower semicontinuity, Hausdorff lower semicontinuity, upper semicontinuity, continuity, Hausdorff continuity closedness.

1. Introduction and preliminaries

Let X, Y, Λ, Γ, M be Hausdorff topological spaces, let Z be a Hausdorff topological vector space, $A \subseteq X$ and $B \subseteq Y$ be nonempty sets. Let $K_1 : A \times \Lambda \to 2^A$, $K_2 : A \times \Lambda \to 2^A$, $T : A \times A \times \Gamma \to 2^B$, $C : A \times \Lambda \to 2^B$ and $F : A \times B \times A \times M \to 2^Z$ be multifunction with $C(x)$ is closed with nonempty interiors different from Z.

For the sake of simplicity, we adopt the following notations. Letters w, m and s are used for a weak, middle and strong, respectively, kinds of considered problems. For subsets U and V under consideration we adopt the notations.

(u, v) w $U \times V$ means $\forall u \in U, \exists v \in V$,

(u, v) m $U \times V$ means $\exists v \in V, \forall u \in U$,

(u, v) s $U \times V$ means $\forall u \in U, \forall v \in V$,

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$\rho_1(U,V)$ means $U \cap V \neq \varnothing$,

$\rho_2(U,V)$ means $U \subseteq V,(u,v)$,

$(u,v)\, \overline{w} U \times V$ means $\exists u \in U, \forall v \in V$ and similarly for $\overline{m}, \overline{s}$,

$\overline{\rho}_1(U,V)$ means $U \cap V = \varnothing$ and similarly for $\overline{\rho}_2$

Let $\alpha \in \{\text{w, m, s}\}$ and $\overline{\alpha} \in \{\overline{w}, \overline{m}, \overline{s}\}$. We consider the following parametric quasiequilibrium problem (in short, (QEP$_{\alpha\rho}$)).

(QEP$_{\alpha\rho}$): Find $\overline{x} \in K_1(\overline{x},\lambda)$ such that $(y,t)\alpha K_2(\overline{x},\lambda) \times T(\overline{x},y,\gamma)$ satisfying

$$\rho(F(\overline{x},t,y,\mu); C(\overline{x},\lambda)).$$

For each $\lambda \in \Lambda, \gamma \in \Gamma, \mu \in M$, we let $E(\lambda) := \{x \in A \mid x \in K_1(x,\lambda)\}$ and let $\Sigma_{\alpha\rho} : \Lambda \times \Gamma \times M \to 2^A$ be a set-valued mapping such that $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu)$ is the solution set of (QEP$_{\alpha\rho}$), i.e.,

$$\Sigma_{\alpha\rho}(\lambda,\gamma,\mu) = \{\overline{x} \in E(\lambda) \mid (y,t)\alpha K_2(\overline{x},\lambda) \times T(\overline{x},y,\gamma) : \rho(F(\overline{x},t,y,\mu); C(\overline{x},\lambda))\}.$$

Throughout the paper we assume that $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu) \neq \varnothing$ for each (λ,γ,μ) in the neighborhoods $(\lambda_0,\gamma_0,\mu_0) \in \Lambda \times \Gamma \times M$.

Special cases of the problem (QEP$_{\alpha\rho}$) are as follows:

(a) If $T(x,y,\gamma) = \{x\}, \Lambda = \Gamma = M, A = B, X = Y, K_1 = K_2 = K, \rho = \overline{\rho_2}$ and replace $C(x,\lambda)$ by $-\operatorname{int} C(x,\lambda)$, replace F by f be a vector function, then (QEP$_{\alpha\overline{\rho_2}}$) become to (PVQEP) in Kimura-Yao [1].

(PQVEP): Find $\overline{x} \in K(\overline{x},\lambda)$ such that

$f(\overline{x},y,\lambda) \notin -\operatorname{int} C(\overline{x},\lambda)$, for all $y \in K(x,\lambda)$.

(b) If $T(x,y,\gamma) = \{x\}, \Gamma = M, A = B, X = Y, K_1(x,\lambda) = K_2(x,\lambda) = K(\lambda), \rho = \overline{\rho_2}$ and replace $C(x,\lambda)$ by $-\operatorname{int} C$, replace F by f be a vector function, then (QEP$_{\alpha\overline{\rho_2}}$) become to (PVEP) in Kimura-Yao [2].

(PVEP): Find $\overline{x} \in K(\lambda)$ such that

$$f(\overline{x},y,\gamma) \notin -\operatorname{int} C, \text{ for all } y \in K(\lambda).$$

(c) If $T(x,y,\gamma) = \{x\}, \Lambda = \Gamma = M, A = B, X = Y, K_1 = K_2 = K$, replace F by f be a vector function, then (QEP$_{\alpha\rho_2}$) becomes (QEP) in Anh-Khanh [3].

(QEP): Find $x \in K(x,\lambda)$ such that

$$f(x,y,\lambda) \in C(x,\lambda), \forall y \in K(x,\lambda).$$

(d) If $T(x,y,\gamma)=\{x\}, \Lambda=\Gamma, A=B, X=Y, K_1=clK, K_2=K, \rho=\rho_1, \rho=\rho_2$ and replace $C(x,\lambda)$ by $Z\setminus -\operatorname{int}C$ with $C\subseteq Z$ be closed and $\operatorname{int}C\neq\emptyset$, then (QEP $_{\alpha\rho_1}$) and (QEP $_{\alpha\rho_2}$) become to (QEP) and (SQEP), respectively in Anh-Khanh [4].

(QEP): Find $\bar{x}\in clK(\bar{x},\lambda)$ such that

$F(\bar{x},y,\lambda)\cap(Z\setminus -\operatorname{int}C)\neq\emptyset,$ for all $y\in K(x,\lambda)$.

and

(SQEP): Find $\bar{x}\in K(\bar{x},\lambda)$ such that

$$F(\bar{x},y,\lambda)\subseteq Z\setminus -\operatorname{int}C, \text{ for all } y\in K(x,\lambda).$$

In this paper we establish sufficient conditions for the solution sets $\Sigma_{\alpha\rho}$ to have the stability properties such as the upper semicontinuity, the lower semicontinuity and the Hausdorff lower semicontinuity, continuity and Hausdorff continuity with respect to parameter λ,γ,μ.

The structure of our paper is as follows. In the remaining part of this section we recall definitions for later uses. Section 2, we establish sufficient conditions for the lower semicontinuity and the Hausdorff lower semicontinuity of solution sets of problems (QEP $_{\alpha\rho}$), and Section 3 is devoted to the upper semicontinuity, continuity and Hausdorff continuity of solution sets of problems (QEP $_{\alpha\rho}$).

Now we recall some notions.

Definition 1.1[5, 6]

Let X and Y be topological vector spaces and $G: X\to 2^Y$ be a multifunction.

(i) G is said to be *lower semicontinuous (lsc) at* $x_0\in X$ if $G(x_0)\cap U\neq\emptyset$ for some open set $U\subseteq Y$ implies the existence of a neighborhood N of x_0 such that $G(x)\cap U\neq\emptyset, \forall x\in N$. G is said to be lower semicontinuous in X if it is lower semicontinuous at each $x_0\in X$.

(ii) G is said to be *upper semicontinuous (usc) at* $x_0\in X$ if for each open set $U\supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U\supseteq G(x), \forall x\in N$. G is said to be upper semicontinuous in X if it is upper semicontinuous at each $x_0\in X$.

(iii) G is said to be *Hausdorff upper semicontinuous (H-usc) at* $x_0\in X$ if for each neighborhood B of the origin in Z, there exists a neighborhood N of x_0 such that, $G(x)\subseteq G(x_0)+B, \forall x\in N$. G is said to be Hausdorff upper semicontinuous in X if it is Hausdorff upper semicontinuous at each $x_0\in X$.

(iv) G is said to be *Hausdorff lower semicontinuous (H-lsc) at* $x_0\in X$ if for each neighborhood B of the origin in Y, there exists a neighborhood N of x_0 such that $G(x_0)\subseteq G(x)+B, \forall x\in N$. G is said to be Hausdorff lower semicontinuous in X if it is Hausdorff lower semicontinuous at each $x_0\in X$.

(v) G is said to be *continuous at* $x_0 \in X$ if it is both lsc and usc at x_0 and to be *H-continuous at* $x_0 \in X$ if it is both H-lsc and H-usc at x_0. G is said to be continuous in X if it is both lsc and usc at each $x_0 \in X$ and to be H-continuous in X if it is both H-lsc and H-usc at each $x_0 \in X$.

(vi) G is said to be *closed at* $x_0 \in X$ if and only if $\forall x_n \to x_0, \forall y_n \to y_0$ such that $y_n \in G(x_n)$, we have $y_0 \in G(x_0)$. G is said to be closed in X if it is closed at each $x_0 \in X$.

Lemma 1.2. ([7, 8]) Let X and Y be topological vector spaces and $G : A \to 2^Y$ *be a multifunction.*

(i) If G is usc at x_0 then G is H-usc at x_0. Conversely if G is H-usc at x_0 and if $G(x_0)$ compact, then G is usc at x_0;

(ii) If G *is H-lsc at* x_0 *then* G *is lsc at* x_0. *The converse is true if* $G(x_0)$ *is compact;*

(iii) If G is usc at x_0 and if $G(x_0)$ is closed, then G is closed at x_0;

(iv) If Z is compact and G is closed at x_0 then G is usc at x_0;

(v) If G has compact values, then G is usc at x_0 if and only if, for each net $\{x_\alpha\} \subseteq X$ which converges to x_0 and for each net $\{y_\alpha\} \subseteq G(x_\alpha)$, there are $y \in G(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$.

2. Lower semicontinuity of solution set

In this section, we discuss the lower semicontinuity and the Hausdorff lower semicontinuity of solution sets for parametric generalized quasiequilibrium problems (QEP$_{\alpha\rho}$).

Theorem 2.1 *Assume for problem (QEP* $_{\alpha\rho}$*) that*

(i) E is lsc at λ_0, K_2 is usc and compact-valued in $K_1(A,\Lambda)\times\{\lambda_0\}$;

(ii) in $K_1(A,\Lambda)\times K_2(K_1(A,\Lambda),\Lambda)\times\{\gamma_0\}$, T *is usc and compact-valued if* $\alpha = s$, *and lsc if* $\alpha = w$ *(or* $\alpha = m$*);*

(iii)the set $\{(x,t,y,\mu,\lambda)\in K_1(A,\Lambda)\times T(K_1(A,\Lambda),K_2(K_1(A,\Lambda),\Lambda),\Gamma)\times K_2(K_1(A,\Lambda),\Lambda)\times\{\mu_0\}\times\{\lambda_0\} : \bar{\rho}(F(x,t,y,\mu);C(x,\lambda))\}$ *is closed.*

Then $\Sigma_{\alpha\rho}$ *is lower semicontinuous at* $(\lambda_0,\gamma_0,\mu_0)$.

Proof. Since $\alpha = \{w,m,s\}$ and $\rho = \{\rho_1,\rho_2\}$, we have in fact six cases. However, the proof techniques are similar. We consider only the cases $\alpha = s, \rho = \rho_2$. Suppose to the contrary that $\Sigma_{s\rho_2}$ is not lsc at $(\lambda_0,\gamma_0,\mu_0)$, i.e., $\exists x_0 \in \Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$, $\exists(\lambda_n,\gamma_n,\mu_n)\to(\lambda_0,\gamma_0,\mu_0)$, $\forall x_n \in \Sigma_{s\rho_2}(\lambda_n,\gamma_n,\mu_n), x_n \not\to x_0$. Since E is lsc at λ_0, there is a net $x_n' \in E(\lambda_n)$, $x_n' \to x_0$. By the above contradiction assumption, there must be a subnet x'_m of x'_n such that, $\forall m$, $x'_m \notin \Sigma_{s\rho_2}(\lambda_m, \gamma_m, \mu_m)$, i.e., $\exists y_m \in K_2(x'_m, \lambda_m), \exists t_m \in T(x'_m, y_m, \gamma_m)$ such that

$$F(x'_m, t_m, y_m, \mu_m) \not\subseteq C(x'_m, \lambda_m). \quad (2.1)$$

As K_2 is usc at (x_0, λ_0) and $K_2(x_0, \lambda_0)$ is compact, one has $y_0 \in K_2(x_0, \lambda_0)$ such that $y_m \to y_0$ (taking a subnet if necessary). By the lower semicontinuity of T at (x_0, y_0, γ_0), one has $t_m \in T(x_m, y_m, \gamma_m)$ such that $t_m \to t_0$. Since $(x'_m, t_m, y_m, \lambda_m, \gamma_m, \mu_m) \to (x_0, t_0, y_0, \lambda_0, \gamma_0, \mu_0)$ and by condition (iii) and (2.1) yields that

$$F(x_0, t_0, y_0, \mu_0) \not\subseteq C(x_0, \lambda_0).$$

which is impossible since $x_0 \in \Sigma_{s\rho_2}(\lambda_0, \gamma_0, \mu_0)$. Therefore, $\Sigma_{s\rho_2}$ is lsc at $(\lambda_0, \gamma_0, \mu_0)$. □

The following example shows that the lower semicontinuity of E is essential.

Example .2.1. Let $A = B = X = Y = \mathbb{R}, \Lambda = \Gamma = M = [0,1], \lambda_0 = 0, C(x,\lambda) = [0,\infty)$, $F(x,t,y,\lambda) = 2^{\lambda}, T(x,y,\lambda) = \{x\}, K_2(x,\lambda) = [0,1]$ and

$$K_1(x,\lambda) = \begin{cases} [-1, 1] & \text{if } \lambda = 0, \\ [-\lambda - 1, 0] & otherwise. \end{cases}$$

We have $E(0) = [-1,1], \forall \lambda \in (0,1]$, $E(\lambda) = [-\lambda - 1, 0], \forall \lambda \in (0,1]$. Hence K_2 is usc and the conditions (ii) and (iii) of Theorem 2.1 are easily seen to be fulfilled. But $\Sigma_{\alpha\rho}$ is not upper semicontinuous at $\lambda_0 = 0$. The reason is that E is not lower semicontinuous. In fact $\Sigma_{\alpha\rho}(0,0,0) = [-1,1]$ and $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu) = [-\lambda - 1, 0], \forall \lambda \in (0,1]$.

The following example shows that in this the special case, assumption (iii) of Theorem 2.1 may be satistied even in cases, but both assumptions (ii$_1$) and (iii$_1$) of Theorem 2.1 in Anh-Khanh [4] are not fulfilled.

Example 2.2. Let $A, B, X, Y, T, \Lambda, \Gamma, M, \lambda_0, C$ as in Example 2.1, and let $K_1(x,\lambda) = K_2(x,\lambda) = [0,1]$ and

$$F(x,y,\lambda) = \begin{cases} [-4, 0] & \text{if } \lambda = 0, \\ [-1 - \lambda, 0] & otherwise. \end{cases}$$

We shows that the assumptions (i), (ii) and (iii) of Theorem 2.1 are satisfied and $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu)) = [0,1], \forall \lambda \in [0,1]$. But both assumptions (ii$_1$) and (iii$_1$) of Theorem 2.1 in Anh-Khanh [4] are not fulfilled.

The following example shows that in this the special case, assumption of Theorem 2.1 may be satistied, but Theorem 2.1 and Theorem 2.3 in Anh-Khanh [4] are not fulfilled.

Example 2.3. Let $A, B, X, Y, T, \Lambda, \Gamma, M, \lambda_0, C$ as in Example 2.2 and let

$K_1(x,\lambda) = K_2(x,\lambda) = [0, \frac{\lambda}{2}]$ and

$$F(x,t,y,\lambda)=\begin{cases}[0,1] & \text{if } \lambda=0,\\ [2,4] & otherwise.\end{cases}$$

We show that the assumptions (i), (ii) and (iii) of Theorem 2.1 are satisfied and $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu))=[0,\frac{\lambda}{2}],\forall\lambda\in[0,1]$. Theorem 2.1 and Theorem 2.3 in Anh-Khanh [4] are not fulfilled. The reason is that F is neither usc nor lsc at $(x,y,0)$.

Remark 2.7. In cases as in Section 1 (a), (b) and (c). Then, Theorem 5.1, 5.2 and 5.3 in [1] Theorem 5.1, 5.2, 5.3 and 5.4 in [2], Theorem 3.1 in [3] are particular cases of Theorem 2.1.

Theorem 2.2. Impose the assumption of Theorem 2.1 and the following additional conditions:

(iv) K_2 *is lsc in* $K_1(A,\Lambda)\times\{\lambda_0\}$ *and* $E(\lambda_0)$ *is compact;*

(v) the set $\{(x,t,y)\in K_1(A,\Lambda)\times T(K_1(A,\Lambda),K_2(K_1(A,\Lambda),\Lambda),\Gamma)\times K_2(K_1(A,\Lambda),\Lambda):\rho(F(x,t,y,\mu_0);C(x,\lambda_0))\}$ *is closed.*

Then $\Sigma_{\alpha\rho}$ *is Hausdorff lower semicontinuous at* $(\lambda_0,\gamma_0,\mu_0)$.

Proof. We consider only for the cases $\alpha=s,\rho=\rho_2$. We first prove that $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ is closed. Indeed, we let $x_n\in\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ such that $x_n\to x_0$. If $x_0\notin\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$, $\exists y_0\in K_2(x_0,\lambda_0),\exists t_0\in T(x_0,y_0,\gamma_0)$ such that

$$F(x_0,t_0,y_0,\mu_0)\not\subseteq C(x_0,\lambda_0). \tag{2.2}$$

By the lower semicontinuity of $K_2(.,\lambda_0)$ at x_0, one has $y_n\in K_2(x_n,\lambda_0)$ such that $y_n\to y_0$. Since $x_n\in\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$, we have

$$F(x_n,t_n,y_n,\mu_0)\subseteq C(x_n,\lambda_0). \tag{2.3}$$

By the condition (v), we see a contradiction between (2,2) and (2.3). Therefore, $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ is closed.

On the other hand, since $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)\subseteq E(\lambda_0)$ and $E(\lambda_0)$ is compact. Hence $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ is compact. Since $\Sigma_{s\rho_2}$ is lower semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$ and $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ is compact. Hence $\Sigma_{s\rho_2}$ is Hausdorff lower semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$. And so we complete the proof. □

3. Upper semicontinuity of solution set

In this section, we discuss the upper semicontinuity, continuity and H-continuity of solution sets for parametric generalized quasiequilibrium problems (QEP$_{\alpha\rho}$).

Theorem 3.1. Assume for problem (QEP$_{\alpha\rho}$) that *(i)* E *is usc at* λ_0 *and* $E(\lambda_0)$ *is compact, and* K_2 *is lsc in* $K_1(A,\Lambda)\times\{\lambda_0\}$*;*

(ii) in $K_1(A,\Lambda)\times K_2(K_1(A,\Lambda),\Lambda)\times\{\gamma_0\}$, T *is usc and compact-valued if* $\alpha = w$ *(or* $\alpha = m$*), and lsc if* $\alpha = s$*;*

(iii) the set $\{(x,t,y,\mu,\lambda)\in K_1(A,\Lambda)\times T(K_1(A,\Lambda),K_2(K_1(A,\Lambda),\Lambda),\Gamma)\times K_2(K_1(A,\Lambda),\Lambda)\times\{\mu_0\}\times\{\lambda_0\}:\rho(F(x,t,y,\mu);C(x,\lambda))\}$ *is closed.*

Then $\Sigma_{\alpha\rho}$ *is both usc and closed at* $(\lambda_0,\gamma_0,\mu_0)$.

Proof. Similar arguments can be applied to six cases. We present only the proof for the cases where $\alpha = w, \rho = \rho_2$. We first prove that $\Sigma_{w\rho_2}$ is upper semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$. Indeed, we suppose to the contrary that $\Sigma_{w\rho_2}$ is not upper semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$, i.e.,there is an open set U of $\Sigma_{w\rho_2}(\lambda_0,\gamma_0,\mu_0)$ such that for all $\{(\lambda_n,\gamma_n,\mu_n)\}$ convergent to $\{(\lambda_0,\gamma_0,\mu_0)\}$, there exists $x_n\in\Sigma_{w\rho_2}(\lambda_n,\gamma_n,\mu_n)$, $x_n\notin U$, $\forall n$. By the upper semicontinuity of E and compactness of $E(\lambda_0)$, one can assume that $x_n\to x_0$ for some $x_0\in E(\lambda_0)$. If $x_0\notin\Sigma_{w\rho_2}(\lambda_0,\gamma_0,\mu_0)$, then $\exists y_0\in K_2(x_0,\lambda_0),\forall t_0\in T(x_0,y_0,\gamma_0)$ such that

$$F(x_0,t_0,y_0,\mu_0)\not\subseteq C(x_0,\lambda_0). \tag{3.1}$$

By the lower semicontinuity of K_2 at (x_0,λ_0), $y_n\in K_2(x_n,\lambda_n)$ such that $y_n\to y_0$. Since $x_n\in\Sigma_{w\rho_2}(\lambda_n,\gamma_n,\mu_n)$, $\exists t_n\in T(x_n,y_n,\gamma_n)$ such that

$$F(x_n,t_n,y_n,\mu_n)\subseteq C(x_n,\lambda_n). \tag{3.2}$$

Since T is usc and $T(x_0,y_0,\gamma_0)$ is compact, one has a subnet $t_m\in T(x_m,y_m,\gamma_m)$ such that $t_m\to t_0$

for some $t_0\in T(x_0,y_0,\gamma_0)$.

By the condition (iii) we see a contradiction between (3.1) and (3.2). Thus, $x_0\in\Sigma_{w\rho_2}(\lambda_0,\gamma_0,\mu_0)\subseteq U$, this contradicts to the fact $x_n\notin U$, $\forall n$. Hence, $\Sigma_{w\rho_2}$ is upper semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$.

Now we prove that $\Sigma_{w\rho_2}$ is closed at $(\lambda_0,\gamma_0,\mu_0)$. Indeed, we suppose that $\Sigma_{w\rho_2}$ is not closed at $(\lambda_0,\gamma_0,\mu_0)$, i.e., there is a net $(x_n,\lambda_n,\gamma_n,\mu_n)\to(x_0,\lambda_0,\gamma_0,\mu_0)$ with $x_n\in\Sigma_{w\rho_2}(\lambda_n,\gamma_n,\mu_n)$ but $x_0\notin\Sigma_{w\rho_2}(\lambda_0,\gamma_0,\mu_0)$. The further argument is the same as above. And so we have $\Sigma_{w\rho_2}$ is closed at $(\lambda_0,\gamma_0,\mu_0)$. □

The following example shows that the upper semicontinuity and compactness of E are essential.

Example 3.1. Let $A=B=X=Y=\mathbb{R},\Lambda=\Gamma=M=[0,1],\lambda_0=0,C(x,\lambda)=[0,\infty)$, $F(x,t,y,\lambda)=2^{\lambda+\sin x},K_1(x,\lambda)=(-\lambda-1,\lambda],K_2(x,\lambda)=\{-1\}$ and $T(x,y,\lambda)=[0,e^{2^x+\cos\lambda}]$.

Then, we have $E(0) = (-1, 0]$ and $E(\lambda) = (-\lambda - 1, \lambda], \forall \lambda \in (0,1]$. We show that K_2 is lsc and assumption (ii) and (iii) of Theorem 3.1 are fulfilled. But $\Sigma_{\alpha\rho}$ is neither usc nor closed at $\lambda_0 = 0$ and $\Sigma_{\alpha\rho}(0,0,0)$ is not compact. The reason is that E is not usc at 0 and $E(0)$ is not compact. In fact $\Sigma_{\alpha\rho}(0,0,0) = (-1,0]$ and $\Sigma_{\alpha\rho}(\lambda, \gamma, \mu) = (-\lambda - 1, \lambda], \forall \lambda \in (0,1]$.

Remark 3.1

(i) In Theorem 4.1 in Kimura-Yao [1] the same conclusion as Theorem 3.1 was proved in anther way. Its assumptions (i)-(iv) derive (i) Theorem 3.1 assumption (v) coincides with (iii) of Theorem 3.1.

(ii) The assumption in Theorem 3.1, we have K_2 is lsc in $K_1(A, \Lambda) \times \{\lambda_0\}$ (which is not imposed in this Theorem 4.1 of [1]). Example 3.2 shows that the lower semicontinuity of K_2 needs to be added to Theorem 4.1 of [1].

Example 3.2. Let $X, Y, \Lambda, \Gamma, M, \lambda_0, C(x, \lambda)$ as in Example 3.1 and let $A = B = [-\frac{1}{2}, \frac{1}{2}]$, $F(x, t, y, \lambda) = x + y + \lambda, K_1(x, \lambda) = [0, \frac{1}{2}], T(x, y, \lambda) = \{x\}$ and

$$K_2(x,\lambda) = \begin{cases} \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\} & \text{if } \lambda = 0, \\ [1, \frac{1}{2}] & otherwise. \end{cases}$$

We have $E(\lambda) = [0,1], \forall \lambda \in [0,1]$. Hence E is usc at 0 and $E(0)$ is compact and condition (ii) and (iii) of Theorem 3.1 are easily seen to be fulfilled. But $\Sigma_{\alpha\rho}$ is not upper semicontinuous at $\lambda_0 = 0$. The reason is that K_2 is not lower semicontinuous.

The following example shows a case where the assumed compactness in Theorem 4.1 of [1] is violated but the assumptions of Theorem 3.1 are fulfilled.

Example 3.3. Let $X, Y, \Lambda, \Gamma, M, T, \lambda_0, C$, as in Example 3.2 and let $A = B = [0, 2)$, $F(x, y, \lambda) = x - y$ and $K_1(x, \lambda) = K_2(x, \lambda) = (x - \lambda - 1] \cap A$. We show that the assumptions of Theorem 3.1 are easily seen to be fulfilled and so $\Sigma_{\alpha\rho}$ is usc and closed at $(0,0,0)$, although A is not compact.

The following example shows that the condition (iii) of Theorem 3.1 is essential.

Example 3.4. Let $\Lambda, \Gamma, M, T, \lambda_0, C$ as in Example 3.2 and let $X = Y = A = B = [0,1]$, $K_1(x, \lambda) = K_2(x, \lambda) = [0,1]$ and

$$F(x, y, \lambda) = \begin{cases} x - y & \text{if } \lambda = 0, \\ y - x & otherwise. \end{cases}$$

We show that assumptions (i) and (ii) of Theorem 3.1 are easily seen to be fulfilled. But $\Sigma_{\alpha\rho}$ is not usc at $\lambda_0 = 0$. The reason is that assumption (iii) is violated.

Indeed, taking $x_n = 0, y_n = 1, \lambda_n = \frac{1}{n} \to 0$ as $n \to \infty$, then $\{(x_n, y_n, \lambda_n\} \to (0,1,0)$ and $F(x_n, y_n, \lambda_n) = F(0,1,1/n) = 1 > 0$ but $F(0,1,0) = -1 < 0$.

The following example shows that all assumptions of Theorem 3.1 are fulfilled. But Theorem 3.2-3.4 in Anh and Khanh [4] cannot be applied.

Example 3.5. Let $A, B, X, Y, \Lambda, \Gamma, M, \lambda_0, C$ as in Example 3.1 and let $K_1(x,\lambda) = K_2(x,\lambda) = [0,3\lambda]$, $T(x,y,\gamma) = [0, 2^{\cos^6 x + \sin^4 x + 2}]$ and

$$F(x,t,y,\lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ e^{\sin^4 x + \cos^2 x + 1} & otherwise. \end{cases}$$

We show that assumptions (i), (ii) and (iii) of Theorem 3.1 are easily seen to be fulfilled. But $\Sigma_{\alpha\rho}$ is usc at $(0,0,0)$. But Theorem 3.2-3.4 in Anh and Khanh [4] cannot be applied. The reason is that F is neither usc nor lsc.

Remark 3.2 In cases as in Section 1 (b). Then, Theorem 4.1 and 4.2 in [2] are particular cases of Theorem 3.1.

Theorem 3.2 Suppose that all conditions in Theorem 2.1 and Theorem 3.1 are satisfied. Then, we have $\Sigma_{\alpha\rho}$ is both continuous and closed at $(\lambda_0, \gamma_0, \mu_0)$.

Theorem 3.3 Suppose that all conditions in Theorem 2.2 and Theorem 3.1 are satisfied. Then, we have $\Sigma_{\alpha\rho}$ is both H-continuous and closed at $(\lambda_0, \gamma_0, \mu_0)$.

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