Dynamic Model of Losses of a Creditor with a Large Mortgage Portfolio

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JEL Classification: G32

Abstract

We propose a dynamic model of mortgage credit losses, which is a generalization of the well-known Vasicek's model of loss distribution. We assume borrowers hold assets covering the instalments and own real estate which serves as collateral. Both the value of the assets and the price of the estate follow general stochastic processes driven by common and individual factors. We describe the correspondence between the common factors and the percentage of defaults, and the loss given default, respectively, and we suggest a procedure of econometric estimation in the model. On an empirical dataset we show that a more accurate estimation of common factors can lead to savings in capital needed to hold against a quantile loss.

1. Introduction

One of the sources of the recent financial crisis was the collapse of the mortgage business. Even if there are ongoing disputes about the causes of the collapse, wrong risk management seems to be one of them. Hence, realistic models of the lending institutions' risk are of great importance.

The textbook approach to the risk control of the loans' portfolio, which is also a part of the IRB standard (Bank for International Settlement, 2006), is that of Vasicek (Vasicek, The Distribution of Loan Portfolio Value, 2002) who deduces the rates of defaults of the borrowers, and consequently the losses of the banks, from the value of the borrowers' assets following a geometric Brownian motion.

In particular, the Vasicek's model assumes that the logarithm of the assets of the *i*-th individual fulfills

$$A_{i,1} = A_{i,0}\exp(\eta + \gamma X_i).$$

Here, $A_{i,0}$ is the individual's wealth at time zero, η and γ are constants, and X_i is a random variable fulfilling

$$X_i = Y + Z_i,$$

where Y is the common factor having a centered normal distribution and $Z_1, Z_2, \ldots$ are i.i.d. centered normal individual factors, independent of Y (Vasicek, Probability of Loss on Loan Portfolio, 1987).

Default of an individual is defined by the state where the value of an individual's assets decreases below a certain thresholdB_i; this threshold is usually interpreted as the sum of the individual's debts (including installments at least). The probability of default is then

$$PD_i = P[A_{i.1} < B_i] = P[X_i < c_i], \quad c_i = \frac{\log B_i - \log A_{i,0} - \eta}{\gamma}.$$

After some calculations (cf. (Vasicek, Probability of Loss on Loan Portfolio, 1987)) we obtain the default rate (DR), defined as

$$DR = \frac{number\ of\ defaults}{number\ of\ loans},$$

approximately fulfilling

$$P[DR \leq x] \doteq N\left(\frac{\left(\sqrt{1-\rho}\right) \cdot N^{-1}(x) - N^{-1}(PD_1)}{\sqrt{\rho}}\right)$$

given a sufficiently large number of loans. Here, N denotes the standard normal cumulative distribution function and

$$\rho = \mathrm{corr}\left(X_i, X_j\right) = \frac{\mathrm{var}(Y)}{\mathrm{var}(Y)+\mathrm{var}(Z_1)}.$$

It follows that the distribution of DR is "heavy-tailed,"[1] with the "heaviness" of the tail dependent on the correlation ρ.

We generalize the Vasicek's model in three ways:

1. We add dynamics to the model (note that the Vasicek's model is only one-period one).
2. We allow more general distribution of the assets. In a nutshell, the main advantage of our model is that asset increments can be described by any continuous distribution, which potentially enables us to use a distribution that is able to fit a particular dataset better than the normal one.
3. We add a sub-model of the losses given default which allows us to calculate the overall percentage loss of the bank.

Similarly as in the Vasicek's paper, in our model, there is a one-to-one correspondence between the common factors and the default rate (DR), and the loss given default (LGD), which allows for econometric estimation of the bivariate series of DR's and LGD's. Thus, these factors can have a general distribution of any kind.

[1] This means that it cannot be successfully approximated by a light-tailed variable.

To our knowledge, only simplified dynamic generalizations of the Vasicek's model incorporating the losses given default have been published (Roesch & Scheule, 2009). However, our approach to the dynamics and/or common modelling of DRs and LGDs is not the only one:

- There are more ways to get the relevant information from the past history of the system, e.g. credit scoring from which the distribution of the DR may be obtained in a standard way (Vasicek, The Distribution of Loan Portfolio Value, 2002) where the distribution of the losses is a function of the probability of default) or observing the credit derivatives (d'Ecclesia, 2008). Another approach to the dynamics could be to track the situation of individual clients (Gupton, Finger, & Bhatia, 1997) or to use affine processes (Duffie, 2005). The usefulness of our approach, however, could lie in the fact that it is applicable "from outside" in the sense that it does not require a bank's internal information.

- Numerous approaches to the joint modeling of DR and the LGD have been published (see e.g. (Witzany J. , 2010), (Yang & Tkachenko, 2012), (Frye, 2000) or (Pykhtin, 2003) and the references therein.) The novelty of our approach, however, is the fact that the form of the dependence of the LGD on the common factor driving the LGD, is not chosen ad-hoc, but it arises naturally from the matter of fact. In particular, it links the LGD to the price of the property serving as a collateral. (Gapko & Šmíd, 2012)

- In its general form, our approach does not assume particular dynamics of the common factors econometric model of which can thus be “plugged” into the model. In contrary to (Gapko & Šmíd, 2012) - a simpler version of our model - multiple generations of debtors are tracked in the presented paper.

Our results show that applying our multi-generational model to a specific dataset leads to a much lower variance in the forecasted credit losses than in the case of the single-generation model. Mainly thanks to the fact that our econometric model uses macroeconomic variables to explain common factors, which is supported by several recent articles, eg (Carling, Jacobson, Lindé, & Roszbach, 2007). It is able to explain changes in risk factors more accurately than a simple model based purely on extraction of common factors from the series of DRs and LGDs. The higher accuracy of the loss forecast then naturally leads to more realistic determination of a quantile loss. In our particular case, the 99.9th quantile loss is lower than in the Vasicek's model.

The paper is organized as follows: after the general definitions (Section 2), where the models of DRs and LGDs are constructed and the procedure of econometric estimation of the model is proposed, Section 3 describes the empirical estimation and finally in Section 4, the paper is concluded.

2. The Model

In the present section, we introduce our model and discuss its estimation. Proofs and some technical details may be found in the Appendix.

2.1 Definition

Let there be (countably) infinitely many potential borrowers. At the time $S_i \in \mathbb{N}_0$, the i-th borrower takes out a mortgage of amount C^i, with help of which, he buys a real property with price $P^i_{S^i} = dC^i$ for some nonrandom $d > 0$. The mortgage is repaid by instalments amounting to $bC^i, b > 0$, at each of the times $S^i + 1, S^i + 2, \dots, S^i + r$, where $r \in \mathbb{N}$ - the duration of the mortgage - is the same for all the borrowers for simplicity.

The assets of the i-th borrower evolve according to stochastic process A^i_t such that, between the times the installments are paid, A follows a Geometrical Brownian Motion with stochastic trend, i.e.

$$A^i_{t-} = A^i_{t-1} \exp\{\Delta Y_t + \Delta Z^i_t\}, \quad t \in \mathbb{N}, \quad t > S^i,$$

where Y_t is a common factor (e.g. a log stock index) and Z_t^i, $\mathbb{E}\Delta Z_t^i = 0$, is a normally distributed individual factor for each $i < n$ with the same variance for each i (Δ stands for a one-period difference).

The instalments are paid by means of selling the necessary amount of the assets, i.e.

$$A_t^i = A_{t-}^i - bC^i, \quad t \in \mathbb{N}, \quad t > S^i.$$

If $A_t^i < 0$ then we say that the borrower defaults at t.

The price P_t^i of the real property serving as a collateral of the mortgage of the i-th debtor fulfils

$$P_t^i = \exp\{\Delta I_t + \Delta E_t^i\} P_{t-1}^i, \quad t > S^i,$$

(recall that $P_{S^i}^i = dC^i$), where I_t is another common factor (e.g. the logarithm of a real estate price index) and $\Delta E_t^i = \mathcal{N}(0, \sigma^2)$ is an individual factor.[2]

The exposure at default H_t^i (i.e. the remaining debt) of the i-th borrower at time t fulfils

$$H_t^i = p(t - S^i)C^i, \quad t > S^i$$

for some decreasing function fulfilling $p(1) = 1, p(\tau) = 0$ if $\tau \leq 0$ or $\tau > r$ (the shape of p may depend on the way of interest calculation and the accounting rules of the bank).

Finally, let

$$\pi_1, \pi_2, \ldots,$$

be the ratios of "newcomers" to the size of the overall portfolio at the times 1, 2, ….

Assume that the increments of the individual factors

$$\Delta X_1^1, \Delta E_1^1, \Delta X_1^2, \Delta E_1^2, \ldots$$

$$\Delta X_2^1, \Delta E_2^1, \Delta X_2^2, \Delta E_2^2, \ldots$$

[2] It would not be difficult to have ΔZ_1^1 and ΔE_1^1 non-normal for the price of loosing closed form formula for function h (see further).

...

are mutually independent and independent of $(Y_t, I_t, \pi_t)_{t\in\mathbb{N}}$ and that, for any *i*, the initial wealth and the size of each mortgage depend, out of all the remaining random variables, only on ω_{S^i}, where

$$\omega_t = (Y_1, I_1, \pi_1, Y_2, I_2, \pi_2, \dots, Y_t, I_t, \pi_t)$$

is the history of the common factors and the percentages of the newcomers up to the start of the mortgage (see (C) in Appendix [sec:Appendix] for details).

Until the end of the Section 2, fix $t \in \mathbb{N}$ and assume that the potential borrowers are numbered so that only those who are active since $t-1$ to t (i.e. those with $t - r \le S^i \le t-1$) and who did not default until $t-1$ are numbered.

2.2 Default rate

Introduce a zero-one variable Q_t^i indicating whether the *i*-th borrower defaults at *t*:

$$Q_t^i = \mathbf{1}\left[A_t^i < 0\right] = \mathbf{1}\left[A_{t-}^i < bC^i\right] = \mathbf{1}\left[a_{t-}^i < b\right] = \mathbf{1}\left[\log a_{t-}^i + \Delta Y_t + \Delta Z_t^i < \log b\right], \qquad (1)$$

where

$$a_t^i = \frac{A_t^i}{C^i}$$

is the value of assets per unit of the mortgage. The first topic of our interest will be the percentage of defaults (i.e., the percentage of the debtors who defaulted at *t*):

$$Q_t \coloneqq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Q_t^i.$$

It is clear from (1) that we may assume, without loss of generality, that $\log b = 0$ (if not than we may subtract $\log b$ from the increments of the common factor). Moreover, we may assume that the variance of ΔZ_t is unit (if not then we could divide $\log a_{t-1}^i$ and ΔY_t^i by its standard deviation).

Thanks to Lemma 8 (see Appendix A.1), we may, similarly to (Vasicek, 2002), apply the Law of Large Numbers to the conditional distribution of Q^i given ω_t to get

$$Q_t = \mathbb{E}(Q_t^1|\omega_t) = \mathbb{P}(Q_t^1 = 1|\omega_t)$$

and compute it, using the Complete Probability Theorem, by formula

$$\mathbb{P}(Q_t^1 = 1|\omega_t) = \sum_{s=t-r}^{t-1} \mathbb{P}(S^1 = s|\omega_t)\mathbb{P}(Q_t^1 = 1|S^1 = s, \omega_t)$$

From the definitions, and thanks to $A(t)$ (see Appendix A.1),

$$\mathbb{P}(Q_t^1 = 1|S^1, \omega_t) = \mathbb{P}\left(\log a_{t-1}^i + \Delta Y_t + \Delta Z_t^1 < 0\middle|S^1, \omega_t\right) = \Psi_t^s(-\Delta Y_t|S^1, \omega_{t-1})$$

where $\Psi_t^s(\cdot\,|s, \omega)$ is the c.d.f. of $\log a_{t-1}^i + \Delta Z_t^i$ given $\omega_{t-1} = \omega,\ S^1 = s$, and because $\mathbb{P}(S^1 = s|\omega_t) = \mathbb{P}(S^1 = s|\omega_{t-1})$ by Lemma 7, we are getting:

Proposition 1

$$Q_t = \sum_{s=t-r}^{t-1} q_{t-1,s}(\omega_{t-1})\Psi_t^s(-\Delta Y_t|s, \omega_{t-1}), \qquad (2)$$

where

$$q_{t-1,s}(\omega_{t-1}) = \mathbb{P}(S^1 = s|\omega_{t-1})$$

♣

Note, that, by Lemma 6 (see Appendix A.1), $\Psi_t^s(\cdot\,|S^1, \omega_{t-1})$ is a strictly increasing c.d.f. of a convolution of two distributions, namely that of $\log a_{t-1}^1$ and the standard normal one. Note also that $q_{t-1,s}$ is in fact the percentage of debts, started at s, and present in the portfolio between times $t-1$ and t.

Corollary 2

For each ω_{t-1}, there exists a one to one mapping between Y_t and Q_t given by (2). In particular,

$$\Delta Y_t = -\Psi_t^{-1}(Q_t|\omega_{t-1}),$$

$$\Psi_t(y,\omega) = \sum_{s=t-r}^{t-1} q_{t-1,s}(\omega)\Psi_t^s(y|s,\omega). \qquad (3)$$

♣

2.3 Loss given default

Since the amount which the bank will recover in case of the default of the i-th debtor at time t is

$$G_t^i = \min(P_t^i, H_t^i)$$
$$= C^i \min\left(d \cdot \exp\left\{\sum_{j=S^i+1}^{t} \left[\Delta I_j + \Delta E_j^i\right]\right\}, p\left(t - S^i\right)\right)$$
$$= C^i \exp\left\{\min\left(d + \sum_{j=S^i+1}^{t} \left[\Delta I_j + \Delta E_j^i\right]\right), \log\left(p\left(t - S^i\right)\right)\right\}$$

we get that the percentage loss given default L_t, i.e. the ratio of the actual losses and the total exposure at default, is

$$L_t = \lim_{n\to\infty} \frac{\sum_{i=1}^n Q_t^i (H^i - G^i)}{\sum_{i=1}^n Q_t^i H^i} = 1 - \lim_{n\to\infty} \frac{\sum_{i=1}^n Q_t^i G^i}{\sum_{i=1}^n Q_t^i H^i}$$

Proposition 3

$$L_t = 1 - \frac{\sum_{s=t-r}^{t-1} \nu_{t,s,\omega_{t-1}} h_{t-s}(\Delta I_{s,t})}{\sum_{s=t-r}^{t-1} \nu_{t,s,\omega_{t-1}}} \qquad \Delta I_{s,t} = I_t - I_s \qquad (4)$$

where

$$\nu_{t,s,\omega} = p(t-s) c_{s,\omega} \Psi_t^s(-\Delta Y_t | s,\omega) q_{t-1,s}(\omega), \qquad c_{s,\omega} = \mathbb{E}(C^1 | S^1 = s,\ \omega_{t-1} = \omega)$$

and

$$h_\tau(\iota) = d\exp\left\{\frac{1}{2}\tau\sigma^2 + \iota\right\} \varphi\left(\frac{\omega_\tau - \iota}{\sqrt{\tau}\sigma} - \sqrt{\tau}\sigma\right) + p(\tau)\left[1 - \varphi\left(\frac{\omega_\tau - \iota}{\sqrt{\tau}\sigma}\right)\right]$$

$$\omega_\tau = \log\left(p(\tau)\right) - \log d,$$

and where φ is the standard normal distribution function. The function h_τ is strictly increasing.

Proof. See appendix A.2

♣

Corollary 4

For given ω_{t-1} there is one-to-one mapping between L_t and I_t, given by (4). *In particular,*

$$I_t = \Upsilon^{-1}_{t,\omega_{t-1}}(1 - L_t), \qquad (5)$$

where

$$\Upsilon_{t,\omega}(\iota) = \frac{1}{\sum_s \upsilon_{t,s,\omega}} \sum_{s=t-r}^{t-1} \nu_{t,s,\omega_t} h_{t-s}\left(\Delta I_{t-1,s} + \iota\right)$$

♣

2.4 Next period

Now, let us proceed to the portfolio at the next period: After renumbering (excluding the defaulted borrowers and adding the newcomers) we get.

Proposition 5

$$q_{t,s}(\omega_t) = \begin{cases} \pi_t & if\ s = t \\ (1 - \pi_t) u_{t,s}(\omega_{t-1}) & if\ t - r < s < t \\ 0 & otherwise \end{cases}$$

where

$$\begin{aligned} u_{t,s}(\omega) &= \mathbb{P}(S^1 = s | Q^1 = 0, \omega_{t-1} = \omega) \\ &= q_{t-1,s}(\omega) \frac{1 - \Psi_t^s(-\Delta Y_t | s, \omega)}{1 - \Psi_t(-\Delta Y_t | \omega) - \left(1 - \Psi_t^s(-\Delta Y_t | t - r, \omega)\right) q_{t,t-r}(\omega)} \end{aligned}$$

and

$$\mathbb{P}[\log a_t^1 \leq z | S^1 = s, \omega_{t-1} = \omega] = \begin{cases} \vartheta_{t,\omega_{t-1}}(z) & if\ S^1 = t \\ \dfrac{\Psi_t^s(z - \Delta Y_t | s, \omega) - \Psi_t^s(-\Delta Y_t | s, \omega)}{1 - \Psi_t^s(-\Delta Y_t | s, \omega)} & otherwise \end{cases}$$

for each $z \geq 0$ *where* $\vartheta_{t,\omega}(z) = \mathbb{P}\left[log\ U^i \leq z |\ \omega_{t-1} = \omega\right], U^i = A_{S_1}$

Proof. See appendix A.3

♣

2.5 Econometrics of the Model

Say we have the sample

$$\pi_1, Q_1, L_1, \pi_2, Q_2, L_2, \dots, \pi_T, Q_T, L_T \qquad (6)$$

at our disposal and want to infer (some of) the parameters of our model, whose complete list is

$$\mathbb{P}\big((X, Y, \pi) \in \cdot\big), c(\cdot), r, d, p(\cdot), \vartheta.(\cdot), \sigma \qquad (7)$$

Clearly, some further simplification of such a rich parameter space has to be done. For simplicity and computability, we decided to postulate values of all the parameters except of $\mathbb{P}\big((X, Y, \pi) \in \cdot\big)$ in the empirical part of our paper so that we are able (recursively) to evaluate the transforming function Ψ_t and Υ_t independently on unknown parameters and the econometrics of the model reduces to the one of the factors *Y* and *I*. In other words, the values of all parameters except of $\mathbb{P}\big((X, Y, \pi) \in \cdot\big)$ were chosen based on empirical observations or expert judgment.

2.6 Numerics of the Model

Generally, Ψ_t is a convolution of truncated (normal) distributions (the defaults are due to the truncations). We chose the Monte Carlo simulation as the easiest way of the functions evaluation which was done in the Mathematica software.

Since the formula for Ψ_t is recursive and involves $\Psi_{t-1}, \dots, \Psi_{t-r}$, which are unknown at the time t, we acted as if the borrowing began at $t = 1$, i.e. we took $q_{1,1} = 1$ and $q_{1,s} = 0$ for all $s < 1$.

3. Empirical estimation

In this part, we describe the estimation procedure of the previously introduced model. The final result of the estimation procedure is a loss distribution and, in particular, a mean predicted loss and a predicted loss quantile on a one-quarter horizon.

The estimation process can be divided into three separate parts: the extraction of both common factors from a historical dataset, a prediction of these factors based on an econometric model and finally, the calculation of future mean and quantile losses given the future values of the factors.

3.1 Data description

We used the same dataset as in (Gapko & Šmíd, 2012), ie, a historical dataset of mortgage delinquencies and started foreclosures, provided by the Mortgage Bankers Association. In our model we took the 90+ delinquency rate at the time t as the default rate, Q_t. Unfortunately, to our knowledge, there is no nationwide public database with banks' losses from mortgage portfolios that could be considered as our loss given default, L_t. Therefore we constructed its proxy by the rate of started foreclosures from the Mortgage Bankers Association and an index of median prices of new homes sold from the US Census Bureau. In particular, because the foreclosures dataset consists of all mortgage loans that fell into the foreclosure process and does not describe how successful the foreclosure process was, we discounted the foreclosures by estimated average values of the collaterals in the portfolio; even if, as we realized, our proxy of the LGD is apparently an ad hoc one, it reflects the fact that the LGD grows with decreasing prices of collaterals.

Formally, we put

$$Q_t = D_t,$$

where D_t is the 90+ delinquency rate at the time t and

$$L_t = \frac{F_t}{D_t J_t},$$

where F_t is the unadjusted rate of started foreclosures from the original dataset and J_t an estimated average value of collaterals in the portfolio calculated as

$$J_t = \sum_{s=t-r}^{t-1} \frac{N_{i,s}}{N_{t-r}} \cdot \frac{\Pi_t}{\Pi_s} \doteq \sum_{s=t-r}^{t-1} q_{t,s}(\omega_{t-1}) \cdot \frac{\Pi_t}{\Pi_s},$$

where $N_{i,s}$ is the number of individuals in the s-th generation at the time t, $\frac{N_{i,t}}{N_t}$ the proportion of individuals of the i-th generation in the whole portfolio at the time t, Π_s the value of the house price index at the time s (recall that we assume unit price of all the collaterals at the start of the mortgage and that $q_{t,s}$ is a function of the observed data).

Both datasets entering our calculations are depicted on the following chart (in percentage of the total outstanding balance).

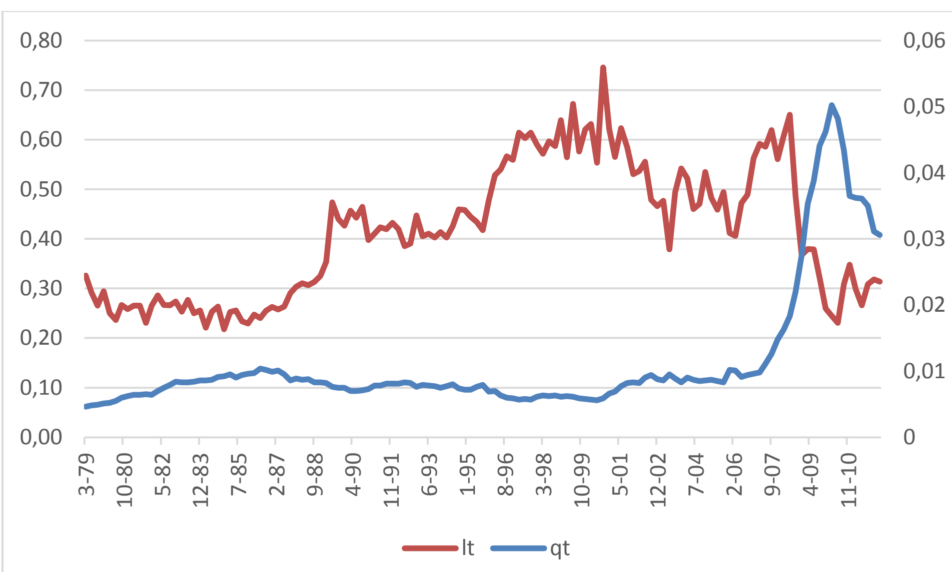

Figure 1: 90+ delinquency rate Q_t and the loss given default L_t

3.2 Choice of Parameters

In order to extract the rate of default and the loss given default, which is the first step in the estimation, we needed to restrict the number of parameters in the extracting functions given by (3) and (5). The parameters

$$c(\cdot), r, d, p(\cdot), \vartheta.(\cdot), \sigma$$

were further postulated as follows:

- The length of the mortgage, r was set to 120 quarters (30 years) based on the long-term average taken from the U.S. Housing Market Conditions survey published quarterly by the U.S. Department of Housing and Urban Development

- The variance of E (the individual factor driving the property price), ie, σ of the distribution with the c.d.f. equal to Φ was set at 0.12 because this value was found to be the one maximizing the log-likelihood in the single-generation model (Gapko & Šmíd, 2012)

- The size of the loan-to-value ratio d at the beginning of the loan is set to 1 (ie, the full mortgage nominal is collateralized by the borrower's property); this is a simplification and a possible point for the model enhancement.

- The quarterly interest rate, which determines the function p, is set to 1%; the function p uses the quarterly simple compounding interest to determine what amount of a mortgage remains to repay

- The standard deviation of each newcoming generation's wealth U_i is assumed to be normal with standard deviation equal to 5

- The parameter c - ie, the expected size of the mortgage, is assumed to be the same for all borrowers

Other parameters, eg, the split on individual generations in a given period, can be calculated directly or derived from our assumptions. For a better understanding of how the original datasets Q and L are translated into the common factors Y and I, resp., we include a comparison of Q and Y (*Figure 2*) and L and I (*Figure 3*). In the *Figures 2* and *3*, the values of the time series I and Q were adjusted to overlap the corresponding time series Y and L, resp. (i.e. Q multiplied by 100 and I multiplied by 10, so that the lines benefit from a single scale representation).

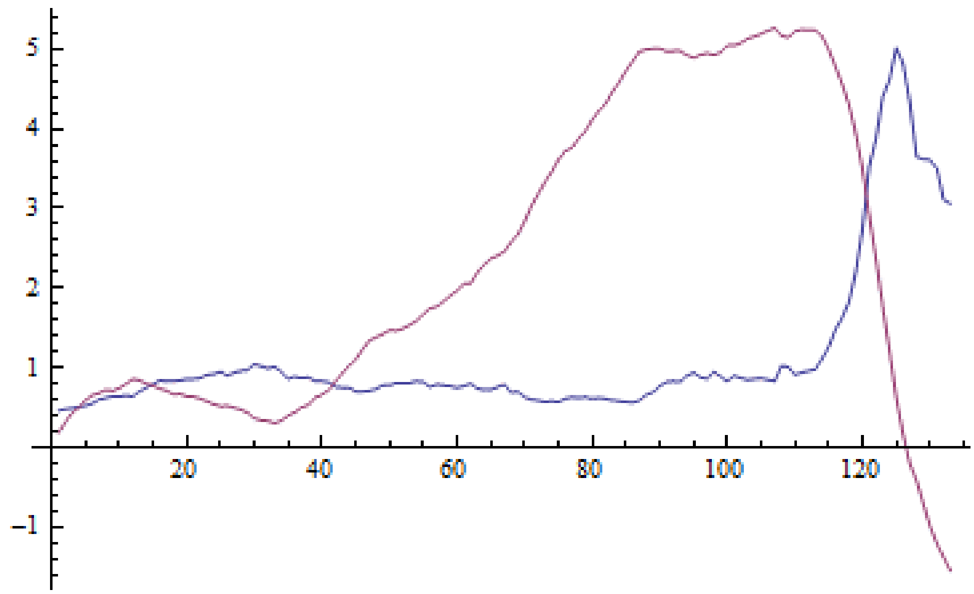

Figure 2: The comparison of Q (blue) and Y (violet)

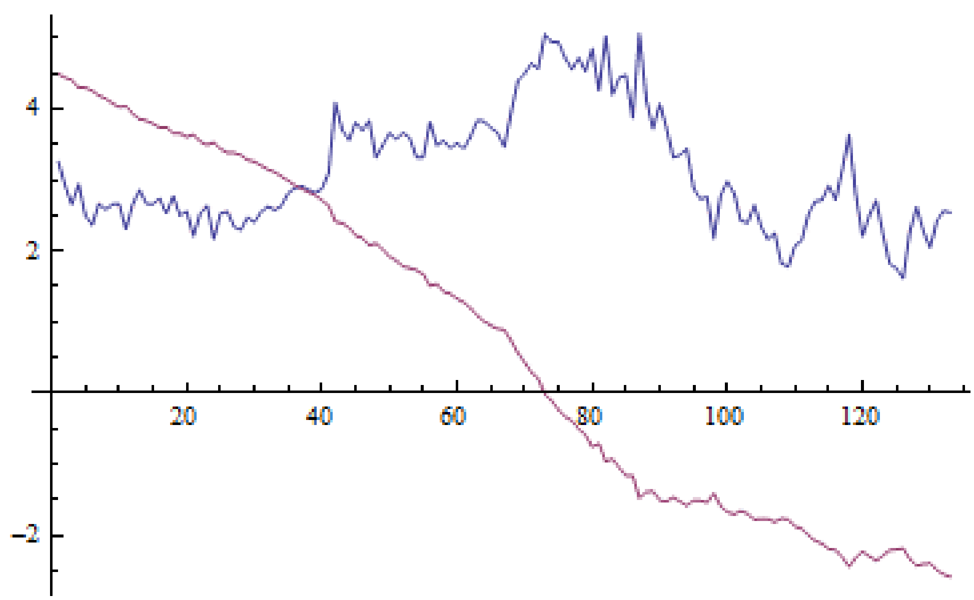

Figure 3: The comparison of *L* (blue) and *I* (violet)

From the beginning of the dataset, there was a sustained growth of house prices, which caused the collateral to exceed the mortgage outstanding amount and thus decreased the LGD. However, in 2007, there was a downturn in housing prices and this is reflected in the increase of the LGD. From the *Figures 2* and *3* we can graphically deduce that the evolution of both common factors might follow some trends, which suggests that there could be a dependence on several macroeconomic variables or stock market indexes. Thus, we chose a Vector Error Correction Model (VECM) with several exogenous macroeconomic variables, namely GDP, unemployment, interest rates, inflation, S&P 500 stock market index and the EUR/USD exchange rate, to capture the joint dynamics of the common factors Y and I. Note that we couldn't use any kind of real estate price index as the LGD values were adjusted by using such an index. Adding it would establish an unsought autocorrelation into the VECM error term.

3.3 Estimation and prediction

The VECM estimation was performed in the Gretl software. First, the stationarity tests of both VECM endogenous variables, ie Y and I, was performed and in both cases, the augmented Dickey-Fuller test rejected the stationarity. The Johansen's cointegration test rejects the absence of the first order cointegration between Y and I on the 10% probability level. Moreover, the first VECM equation, explaining Y, shows that it strongly depends on the year-on-year GDP growth rate. No other macroeconomic variables considered were found significant in this equation, even after lagging them up to four quarters. The second VECM equation, explaining I, also shows dependency on one macroeconomic variable - unemployment rate. Therefore we left the two significant variables, ie, the GDP year-on-year growth rate and the unemployment rate in the model. The following table summarizes our findings. It is obvious that the model is able to explain Y with a much higher predictive power than I, which is probably caused by the fact that changes of I are based on a proxy instead of the actual LGD.

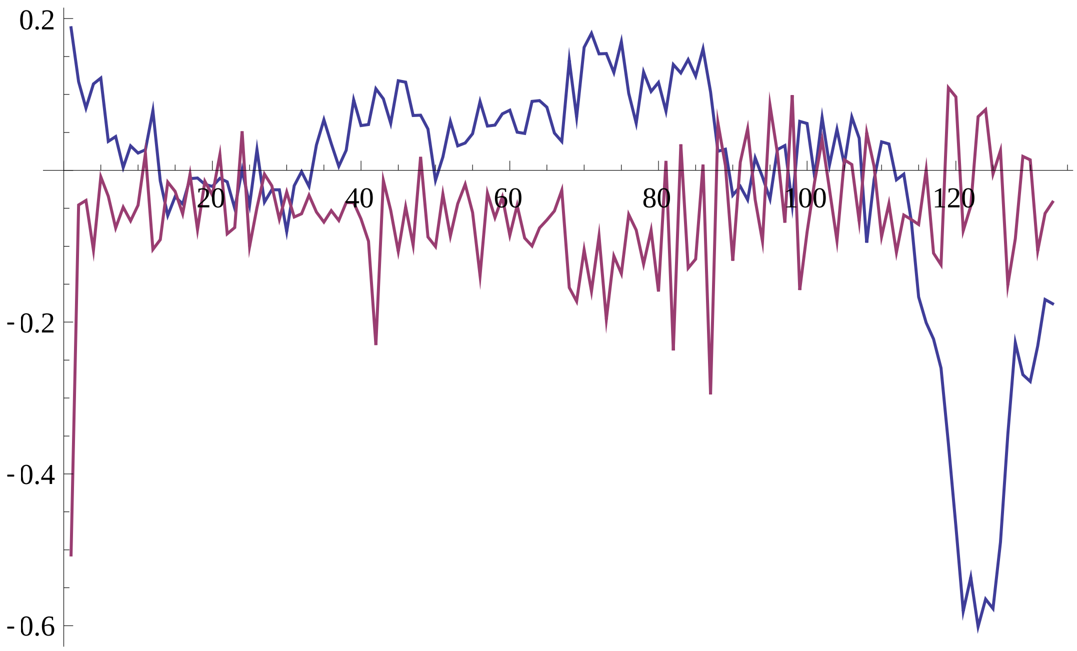

Figure 4: Returns of *Y* (blue) and *I* (violet)

Dependent variable	*Y (s.e.)*	*I (s.e.)*
constant	-0.0098 (0.03)	-0.14*** (0.04)
d1 PD common factor	0.96*** (0.04)	-0.17*** (0.05)
d1 LGD common factor	0.13* (0.07)	-0.24*** (0.09)
GDP year-on-year	0.72*** (0.23)	0.027 (0.3)
Unemployment rate	-0.05 (0.39)	1.07** (0.5)
Error correction term	-0.0067 (0.004)	0.016*** (0.006)
Adjusted R2	91%	15%

Table 1: results of the PD & LGD common factors VECM estimate

Thus the final pair of VECM equations is:

$$Y_t = -0.0098 + 0.96 \cdot dY_t + 0.13 \cdot dI_t + 0.72 \cdot GDP_t - 0.05 \cdot Unemployment_t - 0.0067 \cdot EC_t$$

$$I_t = -0.14 - 0.17 \cdot dY_t - 0.24 \cdot dI_t + 0.027 \cdot GDP_t + 1.07 \cdot Unemployment_t + 0.016 \cdot EC_t$$

We also performed tests of both normality and autocorrelation of residuals. All tests show that error terms of both equations are not autocorrelated and approximately normal.

After the model is estimated, we constructed a prediction of the common factors. To calculate the predicted Y and I, we needed a prediction of exogenous variables in the model, ie, the GDP y/y growth rate and the unemployment rate. As we measured the credit risk only, without an influence of deterioration in economic conditions, we assumed that the unemployment rate stayed for the prediction on its last value and the future GDP change is zero. The following two charts show the development of Y (*Figure 5*) and I (*Figure 6*), including the predicted value.

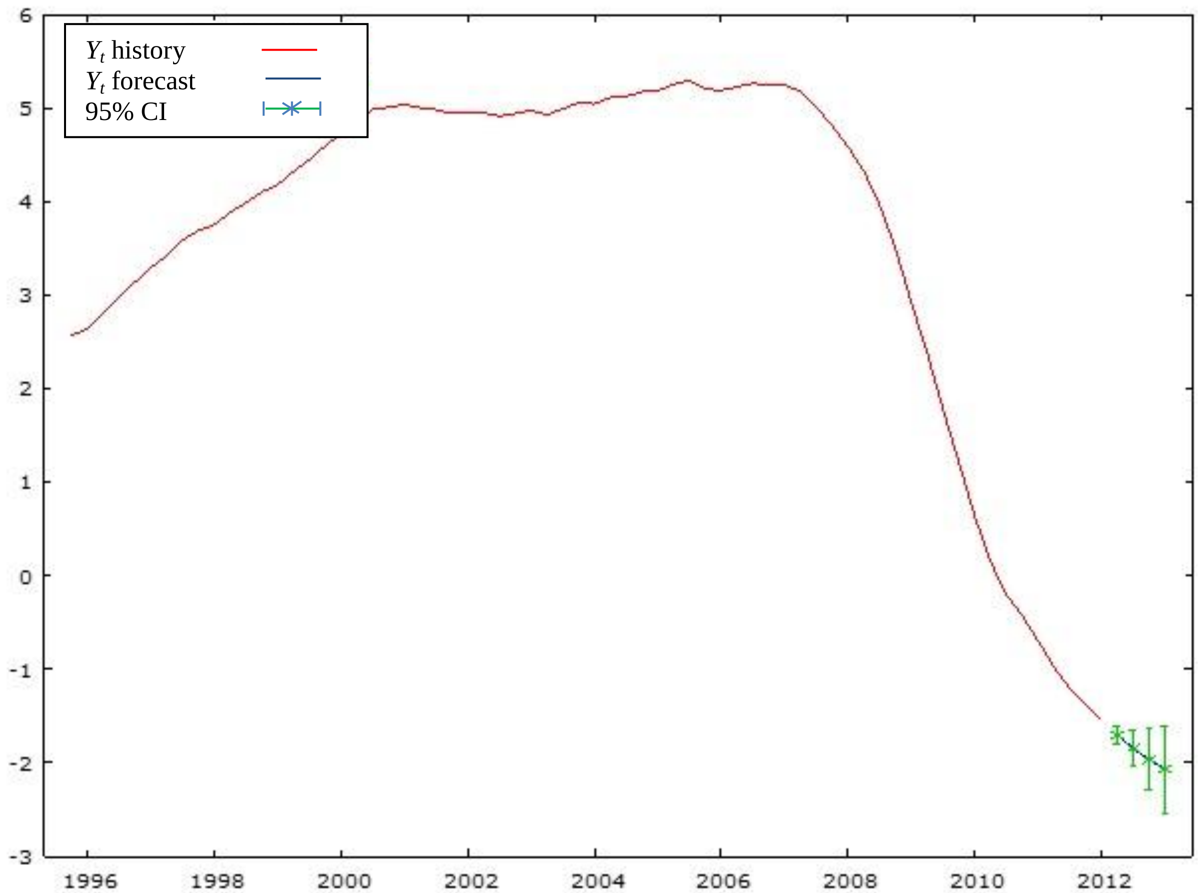

Figure 5: Development of Y with the predicted value (blue) and the prediction standard error (green)

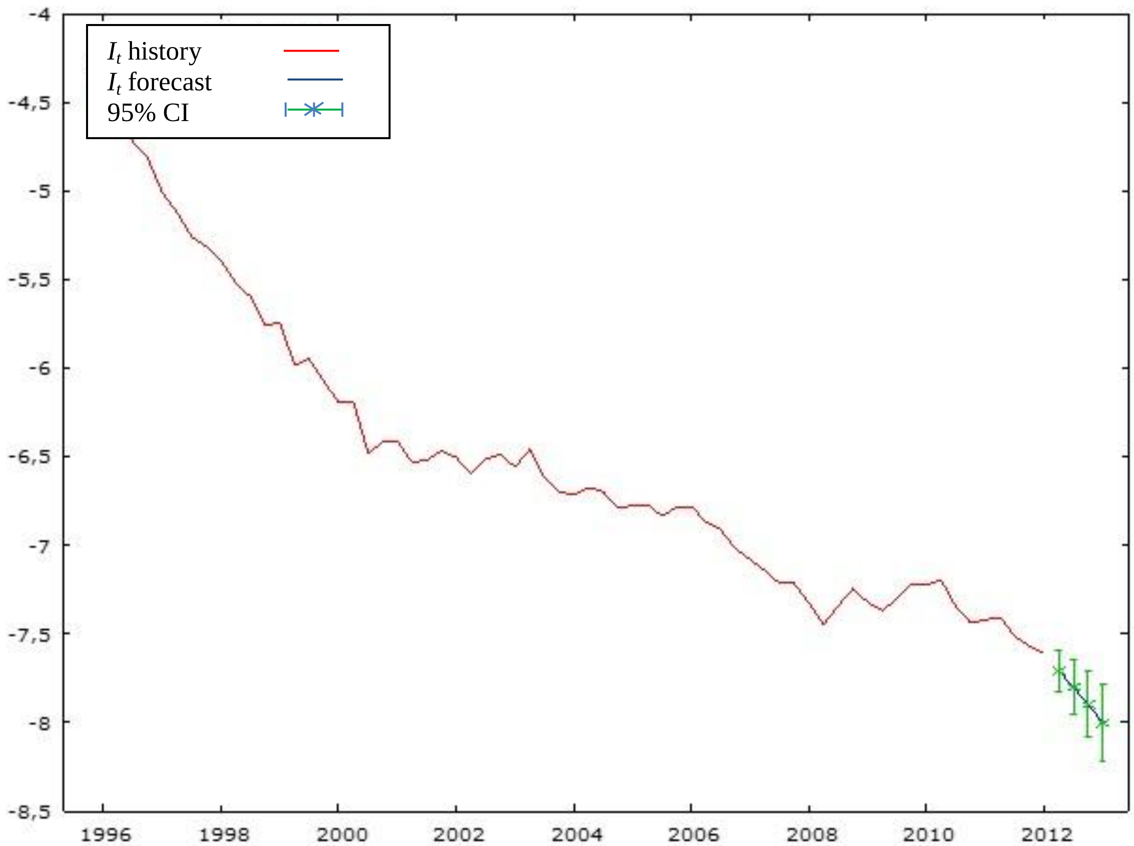

Figure 6: Development of I with the predicted value (blue) and the prediction standard error (green)

3.4 Prediction of losses

The remaining step was to predict a mean and a desired quantile losses. This was done by an inversion function to the factor extraction functions (see (3) and (5)) in the Mathematica software, by which we obtained predicted DR and LGD. These two values were then multiplied to get a loss. The mean loss prediction is quite straightforward as we already have the predicted values of both common factors. However, the quantile loss has to be calculated from the quantile value of both common factors. To be able to compare our quantile loss with the IRB model, we chose to simply calculated the 99.9th quantiles of Q and the 99.9th quantile of L and then multiply them[3]. The calculation of quantiles of Q and L from the quantiles of Y and I was done by the

[3] The 99.9th was chosen to reflect the IRB, which calculates the capital requirement for credit risk as a difference between the mean (expected) loss and the 99.9th quantile loss. Usually, the 99.9th quantile loss is interpreted as a multiplication of the 99.9th quantile of Q and a “downturn” LGD (usually calculated as a 95th quantile of L).

function (2) for Q and by (4) for L. Quantiles of common factors were obtained from their prediction standard error and the assumption that error terms of both VECM equations (see *Table 1*) are normally distributed. (Recall that we were not able to reject the normality). Thus,

$$Y_{q(0.999)} = Y_{t+i} + \sigma_Y \cdot N(0.999) \quad and$$
$$I_{q(0.999)} = I_{t+i} + \sigma_I \cdot N(0.95),$$

where $Y_{q(0.999)}$ and $I_{q(0.999)}$ are 99.9th quantiles of the factors Y and I, resp., Y_{t+i} and I_{t+i} are the common factors predictions, σ_Y and σ_I the regression standard errors and $N(0.999)$ and $N(0.95)$ the 99.9th and the 95th quantile of the standard normal distribution, resp. We constructed a one-quarter quantile loss prediction.

Because the Basel II IRB method calculates a twelve month forward quantile loss, to get a one quarter loss we divided the PD input (last DR value) by two (because the debtor's assets are assumed in the IRB model to be normally distributed, the quarterly PD is exactly one half of the one-year PD, according to the convolution of the normal distribution). We used just one quarter for all the predictions. Both the comparison of the predictions of mean losses calculated by our proposed model and the IRB, and the comparison of the predictions of quantile loss are summarized in the *Table 2*.

Model	***Our***	***IRB***
mean loss	0.84%	0.78%
99.9th quantile loss	1.23%	3.75%

Table 2: comparison of our model's and IRB losses

For the IRB model we have used the last value of default rate as an input for the PD and the last value of our adjusted LGD time series for an LGD. The difference between the IRB and our model computations is that the IRB treats LGD as a fixed variable, whereas in our proposed approach, we constructed a model for LGD predictions. As we can see from *Table 2*, our model predicts much lower quantile loss. This is due to the fact that the explanation of the development of default rates and LGD by our model is much neater than a crude ad-hoc approach of the IRB and thus the standard deviation of loss is lower.

4. Conclusion

In the present paper, we suggested an estimable model of credit losses. The model is based on the assumption of underlying factors that are driving the probability of default and the loss given default. The two novelties of our approach are the multigenerational dimension of the model and the estimated relationship between underlying factors and a macroeconomic environment.

The empirical estimation shows that the model leads to more accurate predictions of future mean and quantile losses than in the Vasicek's framework. This might lead to a saving in the amount of capital that is needed to cover the quantile loss.

Even if the model is rather general and thus a bit more complicated to estimate due to the number of parameters, a bit less could be assumed if a user wished it, especially

- The distribution of the individual factors need not be the same in all periods but it might depend on the time and on the past of the common factor
- A dependence of the individual factors ΔE_t^i and ΔZ_t^i could be established

While the first generalization would not change our formulas much (some indexes would have to be added to the present notation) the second one would bring the necessity to work with a conditional distribution of ΔE given not defaulting, for which no analytical formula exists, even in the simple case of normal factors.

Appendix

A.1 Definitions and Auxiliary Results

First, we have to take into account that the borrowers have to be renumbered in each period in order to remove those who defaulted or fully repaid their mortgage and add those who came newly. Let us assume that the renumbering at t is done as follows: once the indexes $1, 2, \ldots, i-1$ are assigned, a random variable D_t^i is drawn from the Bernoulli distribution with parameter π_t. The index i is consequently given to a newcomer, if $D_t^i = 1$ or to the first unindexed borrower who did not default at t and does not repay fully his mortgage at t, if $D_t^j = 0$. Let us denote S_t^i the starting time of the debtor, indexed by i at t.

Now, denote,

$$\Omega_0 = (\dot{a}_0^1, S_0^1, \dot{a}_0^2, S_0^2, \ldots)$$

and

$$\Omega_t = (Y_\tau, I_\tau, Z_\tau^1, E_\tau^1, Z_\tau^2, E_\tau^2, \ldots)_{\tau \le t}$$

for $t > 0$ and note that, as the distribution of D_t^i depends only on π_t, which itself is a part of the vector ω_t, we have that D_t^i is conditionally independent of $\Omega_t, D_t^1, D_t^2, \ldots, D_t^{i-1}, D_t^{i+1}, \ldots$ given ω_t.

Further, we have to formulate rigorously the assumptions concerning the distribution of the initial wealth and the property price. In particular, we assume that, for each i, $\left(C^i, A_{S_i}^i\right) = \left(C_{i,S_i}, U_{i,S_i}\right)$, where

C

for any i and t, $\left(C_{i,t}, U_{i,t}\right)$ is conditionally independent of $\Omega_t, \left(C_{j,t}, U_{j,t}\right)_{j \neq i}$ given ω_t, and the conditional distribution of $\left(C_{i,t}, U_{i,t}\right)$ given ω_t equals for all i.

Finally, denote $\omega_\infty = (\Delta Y_\tau, \Delta I_\tau)_{\tau \in \mathbb{N}}$ and assume that

$\boldsymbol{A(0)}$

variables $\dot{a}_0^1, S_0^1, \dot{a}_0^2, S_0^2, \ldots$ are mutually independent and independent of ω_t, Ω_t for any $t > 0$, such that a_0^i has the same strictly increasing continuous conditional c.d.f. given S_i for each i.

Now, let us prove that

Lemma 6

For each $t > 0$ the following is true:

$\boldsymbol{A(t)}$

For any i, $\dot{a}_{t-1}^i, \ldots$ is conditionally independent of $\omega_\infty, \left(S_{t-1}^j, a_{t-1}^j\right)_{j\neq i}$ given S_{t-1}^i, ω_{t-1}, such that a_{t-1}^i has the same strictly increasing continuous conditional c.d.f. for each i.

Proof. Let us proceed by induction: For $t = 0$, the assertion follows from $\boldsymbol{A(0)}$. Now, assume $\boldsymbol{A(t)}$ and try to prove $\boldsymbol{A(t+1)}$. Let $i \in \mathbb{N}$. From the basic properties of conditional expectations, we have

$$\mathbb{P}\left(a_t^i < x \middle| D_t^i, \omega_\infty, \left(S_t^k\right)_{k\in\mathbb{N}}, \left(a_t^k\right)_{k\neq i}\right)$$
$$= \begin{cases} \mathbb{P}\left(U_t^i < x \middle| D_t^i, \omega_\infty, \left(S_t^k\right)_{k\in\mathbb{N}}, \left(a_t^k\right)_{k\neq i}\right) = \mathbb{P}\left(U_t^i < x \middle| \omega_t\right) & \text{on } \left[D_t^i = 1\right] \\ \mathbb{P}\left(a_{t-}^{J_i} < x \middle| D_t^i, \omega_\infty, \left(S_t^k\right)_{k\in\mathbb{N}}, \left(a_t^k\right)_{k\neq i}\right) = \mathbb{E}\left(\pi(x) \middle| D_t^i, \omega_\infty, \left(S_t^k\right)_{k\in\mathbb{N}}, \left(a_t^k\right)_{k\neq i}\right) & \text{on } \left[D_t^i = 0\right] \end{cases}$$

where

$$\pi(x) = \mathbb{P}\left(a_{t-}^{J_i} < x \middle| \left(D_t^k, S_{t-1}^k, S_t^k\right)_{k\in\mathbb{N}}, \omega_\infty, J^i, \left(a_t^k\right)_{k\neq i}\right)$$

and J^i is the index of the borrower indexed by i at t given the numbering from $t-1$. On the set $[J_i = j]$, we get

$$\pi(x) = \mathbb{P}\left(a_{t-1}^j + \Delta Z_t^j + \Delta Y_t < x \middle| \left(D_t^k, S_{t-1}^k, S_t^k\right)_{k\in\mathbb{N}}, \omega_\infty, J^i, \left(a_t^k\right)_{k\neq i}\right)$$

$$= \mathbb{E}\left(\mathbb{P}\left(a_{t-1}^{j} + \Delta Z_t^j + \Delta Y_t < x \middle| Q_t^j, \left(D_t^k, S_{t-1}^k, \Delta Z_t^k\right)_{k\in\mathbb{N}}, \omega_\infty, J^i, \left(a_{t-1}^k\right)_{k\neq j}\right) \middle| \left(D_t^k, S_{t-1}^k, S_t^k\right)_{k\in\mathbb{N}}, \omega_\infty, \left(a_t^k\right)_{k\neq i}\right)$$

$$= \mathbb{E}\left(\rho(x) \middle| \left(D_t^k, S_{t-1}^k, S_t^k\right)_{k\in\mathbb{N}}, \omega_\infty, J^i, \left(a_t^k\right)_{k\neq i}\right),$$

where

$$\rho(x) = \mathbb{P}\left(a_{t-1}^j + \Delta Z_t^j + \Delta Y_t < x \middle| \mathbf{1}\left[a_{t-1}^j + \Delta Z_t^j + \Delta Y_t < 0\right], S_{t-1}^j, \omega_t\right)$$

(the last “=” is due to $\boldsymbol{A(t)}$) where, by the textbook calculation

$$\rho(x) = \psi(x, S_{t-1}, \omega_{t-1}), \qquad \psi(x, s, \omega) = \frac{\Psi_t^s(x - \Delta Y_t | s, \omega) - \Psi_t^s(-\Delta Y_t | s, \omega)}{1 - \Psi_t^s(-\Delta Y_t | s, \omega)}$$

on the set $M = \left[Q_t^j = 0, S_t^i = S_{t-1}^i\right]$. Now, because $[J_i = j] \subset M$ and $[J_i = j]_{j\in\mathbb{N}}$ cover the set $\left[D_t^i = 0\right]$, we have by Local Property ((Kallenberg, 2002), Lemma 6.2) that

$$\pi\ (x) = \psi\left(x, S_t^i, \omega_{t-1}\right)$$

on $\left[D_t^i = 0\right]$ finally giving

$$\begin{aligned}
&\mathbb{P}\left(a_t^i < x \middle| \omega_\infty, \left(S_t^k\right)_{k\in\mathbb{N}}, \left(a_t^k\right)_{k\neq i}\right) \\
&= \mathbb{E}\left(\mathbb{P}\left(U_t^i < x \middle| \omega_t\right)\mathbf{1}_{[\boldsymbol{D_1}=\mathbf{1}]} + \psi\left(x, S_t^i, \omega_{t-1}\right)\mathbf{1}_{[\boldsymbol{D_1}=\mathbf{0}]} \middle| \omega_\infty, \left(S_t^k\right)_{k\in\mathbb{N}}, \left(a_t^k\right)_{k\neq i}\right) \\
&= \mathbb{E}\left(\mathbb{P}\left(U_t^1 < x | \omega_t\right)\mathbf{1}_{[\boldsymbol{D_t^1}=\mathbf{1}]} + \psi\left(x, S_t^i, \omega_{t-1}\right)\mathbf{1}_{[\boldsymbol{D_t^1}=\mathbf{0}]} \middle| \omega_t\right),
\end{aligned} \tag{8}$$

where the last "=” is due to the conditional independence of D_t of Ω_t, hence $\boldsymbol{A(t+1)}$ is proved.

♣

Lemma 7

For any $i \in N$, S_t^i is conditionally independent of $\omega_\infty, (S_t^{\dot{j}})_{J\neq \iota}$, given ω_t.

Proof. For $t=0$ the Lemma follows from $\boldsymbol{A}(\mathbf{0})$. Let $\tau>0$ and let the Lemma holds for $t=\tau-1$. ie,

$$\mathbb{P}\left(S_{\tau-1}^{k}\middle|\left(S_{t}^{j}\right)_{j\neq k},\dot{\omega}_{\infty}\right)=\mathbb{P}\left(S_{\tau-1}^{k}\middle|\,\dot{\omega}_{\tau-1}\right).$$

By our construction, S_{τ}^{i} is a function of $S_{\tau-1}^{J_i}$ where J_i is defined by the previous proof. Similarly to the previous proof we show that, on $[J_i=j]$ the probability that $S_{\tau}^{i}=s$ given all the variables $a_{\tau-1}^{\cdot}$ depends only on $q_{\tau-1}(\omega_{\tau-1})$ and on π_{τ}.

♣

Lemma 8

$Q_t^1, Q_t^2, \ldots$ are mutually conditionally independent given ω_t.

Proof. It follows from Lemma 6 that Q_t^i is conditionally independent of $\left(Q_t^k, S_{t-1}^k\right)_{k\neq i}$ given $\left(\omega_t, S_{t-1}^i\right)$. Thanks to Lemma 7 and independence of variables $\Delta Z_t^{\cdot}$ we get that S_{t-1}^i is conditionally independent of $\left(Q_t^k, S_{t-1}^k\right)_{k\neq i}$ given ω_t which gives the Lemma by the Chain rule for conditional independence ((Kallenberg, 2002), Proposition 6.8).

A.2 Proof of Proposition 3

By (Kallenberg, 2002), Corollary 5.

$$L_t=1-\frac{\lim\limits_{n\to\infty} n^{-1}\sum_{i=1}^{n}Q_t^iG_t^i}{\lim\limits_{n\to\infty} n^{-1}\sum_{i=1}^{n}Q_t^iH_t^i}.$$

Further, by Lemma 8 and by the independence of variables E, the summands in both sums are conditionally independent given ω_t, hence, by the Law of large numbers,

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}Q_t^iG^i=\mathbb{E}(Q_t^1G_t^1|\omega_t)=\mathbb{E}(\mathbb{E}(Q_t^1G_t^1|\omega_t,S_{t-1}^1)|\omega_t)$$

$$= \sum_{s=t-r}^{t-1} \mathbb{E}(Q_t^1 G_t^1 | \omega_t, S_{t-1}^1 = s) q_{t-1,s}(\omega_{t-1})$$

$$= \sum_{s=t-r}^{t-1} \mathbb{E}(G_t^1 | \omega_t, S_{t-1}^1 = s) \mathbb{E}(Q_t^1 | \omega_t, S_{t-1}^1 = s) q_{t-1,s}(\omega_{t-1})$$

$$= \sum_{s=t-r}^{t-1} \mathbb{E}(G_t^1 | \omega_{t-1}, S_t^1 = s) \mathbb{E}(Q_t^1 | \omega_{t-1}, S_t^1 = s) q_{t-1,s}(\omega_{t-1})$$

$$= \sum_{s=t-r}^{t-1} \upsilon_{t,s,\omega_{t-1}} h_{t-s}\left(\Delta I_{s,t}\right)$$

$$h_r(\iota) = d\mathbb{E}(\exp\{min(\iota + e_r, w_r)\}) \qquad e_r \sim \mathcal{N}(0, r\sigma^2)$$

and analogously,

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} Q_t^i H^i = \sum_{s=t-r}^{t-1} \upsilon_{t,s,\omega_{t-1}}$$

As to h, we are getting

$$\begin{aligned} h_r(\iota) &= d\mathbb{E}(\exp\{\iota\} \exp\{\min(e_r, w_r - \iota)\}) \\ &= de^{\iota} \left[\int_{-\infty}^{w_r-\iota} e^x d\Phi_{(r)}(x) + e^{w_r-\iota} \Big(1 - \Phi_{(r)}(w_r - \iota)\Big) \right] \\ &= de^{\iota} \int_{-\infty}^{w_r-\iota} e^x d\Phi_{(r)}(x) + p(t-s) \Big(1 - \Phi_{(r)}(w_r - \iota)\Big), \end{aligned}$$

where $\Phi_{(\nu)}$ is a c.d.f. $\mathcal{N}(0, \nu\sigma^2)$ - when we put $\varsigma = \sqrt{r}\sigma$, we get

$$\int_{-\infty}^{w_r-\iota} e^x d\Phi_{(r)}(x) = \int_{-\infty}^{w_r-\iota} \frac{1}{\sqrt{2\pi}\varsigma} e^{-\frac{x^2}{2\varsigma^2}} e^x dx$$

$$= \frac{1}{\sqrt{2\pi}\varsigma} \int_{-\infty}^{w_r-\iota} \exp\left\{-\frac{x^2 - 2\varsigma^2 x + \varsigma^4}{2\varsigma^2} + \frac{1}{2}\varsigma^2\right\} dx$$

$$= \exp\left\{\frac{1}{2}\varsigma^2\right\} \int_{-\infty}^{w_r-\iota} \frac{1}{\sqrt{2\pi}\varsigma} \exp\left\{-\frac{(x-\varsigma^2)^2}{2\varsigma^2}\right\} dx$$

$$= \exp\left\{\frac{1}{2}\varsigma^2\right\}\mathbb{P}[N(\varsigma^2,\varsigma^2) < w_r - \iota] = \exp\left\{\frac{1}{2}\varsigma^2\right\}\varphi\left(\frac{w_r - \iota}{\varsigma} - \varsigma\right)$$

hence

$$h_r(\iota) = d\exp\left\{\frac{1}{2}r\sigma^2 + \iota\right\}\varphi\left(\frac{w_r - \iota}{\sqrt{r}\sigma} - \sqrt{r}\sigma\right) + p(r)\left[1 - \varphi\left(\frac{w_r - \iota}{\sqrt{r}\sigma}\right)\right].$$

The monotonicity is proved by the fact that

$$\begin{aligned}\frac{\partial}{\partial\iota}h_r(\iota) &= de^{\iota}\left(\Phi_{(r)}(w_r - \iota)e^{(w_r - \iota)} - \int_{-\infty}^{w_r - \iota}\Phi_{(r)}(x)e^x dx\right)\\ &= de^{\iota}\left(\Phi_{(r)}(w_r - \iota)\int_{-\infty}^{w_r - \iota}e^x dx - \int_{-\infty}^{w_r - \iota}\Phi_{(r)}(x)e^x dx\right)\\ &= de^{\iota}\left(\int_{-\infty}^{w_r - \iota}\Phi_{(r)}(w_r - \iota)e^x dx - \int_{-\infty}^{w_r - \iota}\Phi_{(r)}(x)e^x dx\right)\\ &= de^{\iota}\left(\int_{-\infty}^{w_r - \iota}\left[\Phi_{(r)}(w_r - \iota) - \Phi_{(r)}(x)\right]e^x dx\right) > 0.\end{aligned}$$

A.3 Proof of Proposition 5

The fact that $q_{t,t} = \pi_t$ follows from the definition, as well as the fact that $q_{t,s} = 0$ for $s \leq t - r$.

Let $t - r < s < t$, and let J_i be the previous index of the borrower indexed by i at t (it can be eg, a zero if the borrower is a newcomer). Clearly, $S_t^i = s \Leftrightarrow \mathrm{D}_t^i = 0 \wedge \mathrm{S}_{t-1}^{J_i} = \mathrm{s}$ which implies

$$\begin{aligned}&\mathbb{P}\left(S_t^i = s \middle| \omega_t = \omega\right) = \mathbb{P}\left(D_t^i = 0, S_{t-1}^{J_i} = s \middle| \omega_t = \omega\right)\\ &= (1 - \pi_t)\mathbb{P}\left(S_{t-1}^{J_i} = s \middle| D_t^i = 0, \omega_t = \omega\right)\\ &= (1 - \pi_t)\sum_j \mathbb{P}\left(S_{t-1}^j = s \middle| J_i = j, D_t^i = 0, \omega_t = \omega\right)\mathbb{P}\left(J_i = j \middle| D_t^i = 0, \omega_t = \omega\right).\end{aligned} \tag{9}$$

Further, as

$$J_i = j \Leftrightarrow \sum_{k=1}^{i-1} \left(1 - D_t^k\right) = \sum_{k=1}^{i-1} \mathbf{1}[Q_{t-1}^k = 0, S_{t-1}^k \neq t - r] \wedge Q_t^j = 0 \wedge S_{t-1}^j \neq t - r \wedge D_t^i = 0$$

we have, from the conditional independence

$$\mathbb{P}\left(S_{t-1}^j = s \middle| J_i = j, D_t^i = 0, \omega_t = \omega\right) = \mathbb{P}\left(S_{t-1}^j = s \middle| Q_t^j = 0, S_{t-1}^j \neq t - r, \omega_t = \omega\right)$$
$$= \frac{\mathbb{P}\left(S_{t-1}^j = s, Q_t^j = 0, S_{t-1}^j \neq t - r \middle| \omega_t = \omega\right)}{\mathbb{P}\left(Q_t^j = 0, S_{t-1}^j \neq t - r \middle| \omega_t = \omega\right)}$$
$$= \frac{\mathbb{P}\left(S_{t-1}^j = s, Q_t^j = 0 \middle| \omega_t = \omega\right)}{\mathbb{P}\left(Q_t^j = 0 \middle| \omega_t = \omega\right) - \mathbb{P}\left(Q_t^j = 0, S_{t-1}^j = t - r \middle| \omega_t = \omega\right)}$$
$$= \frac{\mathbb{P}\left(Q_t^j = 0 \middle| S_{t-1}^j = s, \omega_t = \omega\right) \mathbb{P}\left(S_{t-1}^j = s \middle| \omega_t = \omega\right)}{\mathbb{P}\left(Q_t^j = 0 \middle| \omega_t = \omega\right) - \mathbb{P}\left(Q_t^j = 0 \middle| S_{t-1}^j = t - r, \omega_t = \omega\right) \mathbb{P}\left(S_{t-1}^j = t - r \middle| \omega_t = \omega\right)}$$
$$= \frac{\left(1 - \Psi_t^s(-\Delta Y_t | s, \omega_{t-1})\right) q_{t-1,s}(\omega_{t-1})}{1 - \Psi_t(-\Delta Y_t | \omega_{t-1}) - \left(1 - \Psi_t^s(-\Delta Y_t | t - r, \omega_{t-1})\right) q_{t-1,t-r}(\omega_{t-1})},$$

which, not being dependent on j, may be pulled out from the sum in (9).

The formula for a_t is proved similarly to (8).

Compliance with Ethical Standards

Hereby the authors confirm that there they are not aware of any conflict of interests. The paper is an original manuscript, which is not published, nor considered to be published in any other journal. Also, authors confirm that they did not manipulate data in any other way except those described in the manuscript.

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